SECTION 3
MECHANICAL ENGINEERING PRINCIPLES

Section Editor: Mark Davies

PART 1

STATICS OF RIGID BODIES

Robert Paine

In general, the study of mechanics may be divided into two distinct areas. These are *statics,* which involves the study of bodies at rest, and *dynamics,* which is the study of bodies in motion. In each case it is important to select an appropriate mathematical model from which a "free body diagram" may be drawn, representing the system in space, with all the relevant forces acting on that system.

When a set of forces acts on a body, it gives rise to a resultant force or moment or a combination of both. The situation may be determined by considering three mutually perpendicular directions on the free body diagram and resolving the forces and moment in these directions. If the three directions are denoted by x, y, and z, then the sum of forces may be represented by ΣF_x, ΣF_y, and ΣF_z and the sum of the moments about respective axes by ΣM_x, ΣM_y, and ΣM_z. Then for equilibrium the following conditions must hold:

$$\Sigma F_x = \Sigma F_y = \Sigma F_z = 0 \tag{3.1}$$

$$\Sigma M_x = \Sigma M_y = \Sigma M_z = 0 \tag{3.2}$$

If the conditions in equations (3.1) and (3.2) are not satisfied, then there is a resultant force or moment, which is given by

$$F = [(\Sigma F_x)^2 + (\Sigma F_y)^2 + (\Sigma F_z)^2]^{1/2}$$

$$M = [(\Sigma M_x)^2 + (\Sigma M_y)^2 + (\Sigma M_z)^2]^{1/2}$$

The six conditions given in equations (3.1) and (3.2) satisfy problems in three dimensions. If one of these dimensions is not present (say, the z direction) the system reduces to a set of coplanar forces, and then

$$\Sigma F_x = \Sigma M_x = \Sigma M_y = 0$$

are automatically satisfied, and the necessary conditions of equilibrium in a two-dimensional system are

$$\Sigma F_x = \Sigma F_y = \Sigma M_z = 0 \tag{3.3}$$

If the conditions in equation (1.3) are not satisfied, then the resultant force or moment is given by

$$F = [(\Sigma F_x)^2 + (\Sigma F_y)^2]^{1/2}$$

$$M = \Sigma M_z$$

The above equations give solutions to what are said to be "statically determinate" systems. These are systems where there are the minimum number of constraints to maintain equilibrium (Mathesun 1987).

PART 2

STRENGTH OF MATERIALS

Robert Paine

Weight: The weight (W) of a body is that force exerted due to gravitational attraction on the mass (m) of the body: $W = mg$, where g is the acceleration due to gravity.

Center of gravity: This is a point, which may or may not be within the body, at which the total weight of the body may be considered to act as a single force. The position of the center of gravity may be found experimentally or by analysis. When using analysis the moment of each element of weight, within the body, about a fixed axis is equated to the moment of the complete weight about that axis:

$$\tilde{x} = \Sigma \delta mg \cdot x / \Sigma \delta mg,\ \tilde{y} = \Sigma \delta mg \cdot y / \Sigma \delta mg$$

$$\tilde{z} = \Sigma \delta mg \cdot z / \Sigma \delta mg$$

where δm is an element of mass at a distance of x, y or z from the respective axis, and $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$ are the positions of the centers of gravity from these axes. Table 3.1 shows the position of the center of gravity for some standard shapes. (See Meriam and Kraige 1987 for a more comprehensive list.)

Shear force and bending moment: If a beam subject to loading, as shown in Figure 3.1, is cut, then in order to maintain equilibrium a shear force (Q) and a bending moment (M) must be applied to each portion of the beam. The magnitudes of Q and M vary with the type of loading and the position along the beam and are directly related to the stresses and deflections in the beam.

Relationship between shearforce and bending moment: If an element of a beam is subjected to a load w then the following relationship holds:

$$\frac{d^2M}{dx^2} = \frac{dF}{dx} = -w$$

Table 3.2 shows examples of bending moments, shear force, and maximum deflection for standard beams.

Bending equation: If a beam has two axes of symmetry in the xy plane then the following equation holds:

$$M_Z / I_Z = E / R_Z = \sigma / y$$

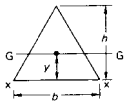
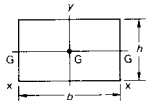
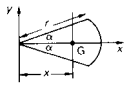
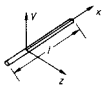
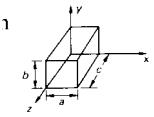
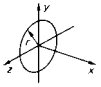
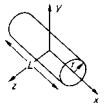
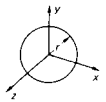
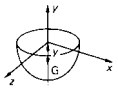
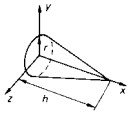
where M_Z is the bending moment, R_z the radius of curvature, I_z the moment of inertia, E the modulus of elasticity, y the distance from the principal axis, and σ the stress.

Torsion equation: If a circular shaft is subject to a torque (T) then the following equation holds:

$$T / J = \tau / r = G\theta / L$$

where J is the polar second moment of area, G the shear modulus, L the length, θ the angle of twist, τ the shear stress, and r the radius of the shaft.

TABLE 3.1 Centers of Gravity and Moments of Inertia or Second Moments of Area for Two-Dimensional Figures

Shape		G	I
Triangular area		$\bar{y} = h/3$	$I_{GG} = bh^3/36$ $I_{XX} = bh^3/12$
Rectangular-area			$I_{GG} = bh^3/12$ $I_{XX} = bh^3/3$
Circular sector		$\bar{x} = \frac{2r}{3}\frac{\sin\alpha}{\alpha}$	$I_{XX} = \frac{r^4}{4}\left(\alpha - \frac{1}{2}\sin^2\alpha\right)$ $I_{YY} = \frac{r^4}{4}\left(\alpha + \frac{1}{2}\sin^2\alpha\right)$
Slender rod			$I_{XX} = I_{ZZ} = ml^2/12$
Rectangular prism			$I_{XX} = m(b^2 + c^2)/12$ $I_{YY} = m(c^2 + a^2)/12$ $I_{ZZ} = m(a^2 + b^2)/12$
Thin disk			$I_{XX} = mr^2/2$ $I_{YY} = I_{ZZ} = mr^2/4$
Circular cylinder			$I_{XX} = mr^2/2$ $I_{YY} = I_{ZZ} = m(3r^2 + L^2)/12$
Sphere			$I_{XX} = I_{YY} = I_{ZZ} = 2mr^2/5$
Hemisphere		$\bar{y} = 3r/8$	$I_{YY} = 2mr^2/5$
Circular cone		$\bar{x} = h/4$	$I_{XX} = 3mr^2/10$ $I_{YY} = \frac{3mr^2}{20} + \frac{mh^2}{10}$

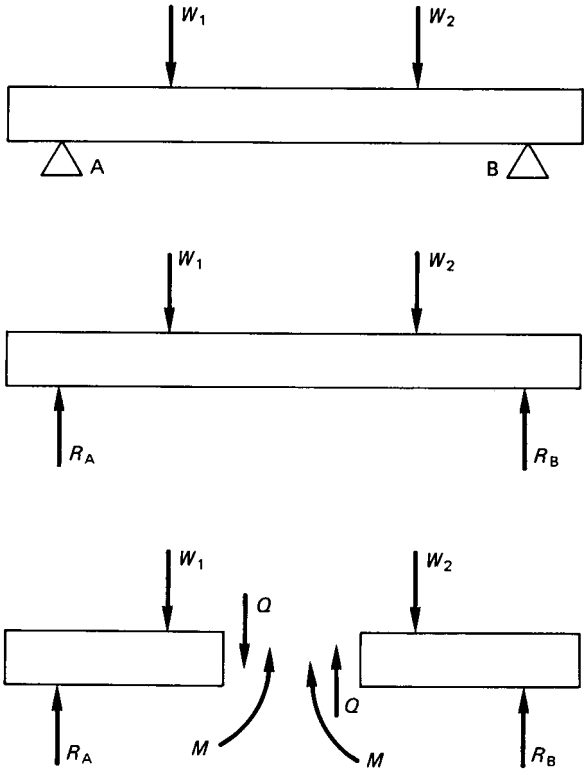
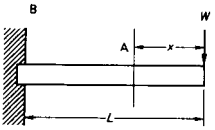
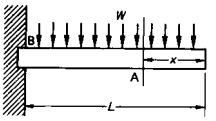


FIGURE 3.1

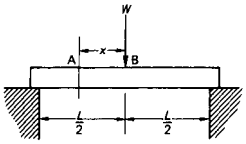
TABLE 3.2



One concentrated load W
M at $A = Wx$, Q at $A = W$
M greatest at B, and $= WL$
Q uniform throughout
Maximum deflection $= WL^3/3EI$ at the free end.



Uniform load of W
M at $A = Wx^2/2L$
Q at $A = Wx/L$
M greatest at $B = WL/2$
Q greatest at $B = W$
Maximum deflection $= WL^3/8EI$ at the free end.

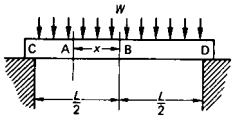


One concentrated load at the center of a beam

$$M \text{ at } A = \frac{W}{2}\left(\frac{L}{2} - x\right),$$

Q at $A = W/2$

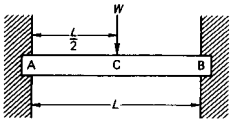
M greatest at $B = WL/4$
Q uniform throughout
Maximum deflection $= WL^3/48EI$ at the center



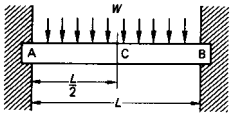
Uniform load W

$$M \text{ at } A = \frac{W}{2L}\left(\frac{L^2}{4} - x^2\right)$$

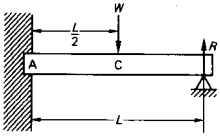
Q at $A = Wx/L$
M greatest at $B = WL/8$
Q greatest at C and $D = W/2$ maximum deflection at $B = 5WL^3/384EI$



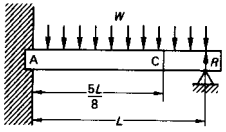
Beam fixed at ends and loaded at center.
M is maximum at A, B, and C and $= WL/8$.
Maximum deflection at $C = WL^3/192EI$



Beam fixed at ends with uniform load.
M maximum at A and B and $= WL/12$
Maximum deflection at $C = WL^3/384EI$



One concentrated load W
Reaction $R = 5W/16$
M maximum at A, and $= 3WL/16$
M at $C = 5WL/32$
Maximum deflection is $L/\sqrt{5}$ from the free end, and $= WL^3/107EI$



Uniform load W
Reaction $R = 3W/8$
M maximum at A, and $= WL/8$
M at $C = 9WL/128$
Maximum deflection is $3L/8$ from the free end, and $= WL^2/187EI$

PART 3

DYNAMICS OF RIGID BODIES

Robert Paine

3.1 BASIC DEFINITIONS

Newton's Laws of Motion

First Law. A particle remains at rest or continues to move in a straight line with a constant velocity unless acted on by an external force.

Second Law. The sum of all the external forces acting on a particle is proportional to the rate of change of momentum.

Third Law. The forces of action and reaction between interacting bodies are equal in magnitude and opposite in direction.

Newton's law of gravitation, which governs the mutual interaction between bodies, states:

$$F = Gm_1m_2/x^2$$

where F is the mutual force of attraction, G is a universal constant called the constant of gravitation, which has a value 6.673×10^{-11} m^3 kg^{-1} s^2, m_1 and m_2 are the masses of the two bodies, and x is the distance between the centers of the bodies.

Mass (m) is a measure of the amount of matter present in a body.

Velocity is the rate of change of distance (x) with time (t):

$$v = dx/dt \text{ or } \dot{x}$$

Acceleration is the rate of change of velocity (v) with time (t):

$$a = dv/dt \text{ or } d^2x/dt^2 \text{ or } \ddot{x}$$

Momentum is the product of the mass and the velocity. If no external forces are present then the momentum of any system remains constant. This is known as the conservation of momentum.

Force is equal to the rate of change of momentum (mv) with time (t):

$$F = d(mv)/dt$$

$$F = m \cdot dv/dt + v \cdot dm/dt$$

If the mass remains constant then this simplifies to $F = m \cdot dv/dt$, i.e., Force = mass $\times$ acceleration, and it is measured in newtons.

Impulse (I) is the product of the force and the time that force acts. Since $I = Ft = mat = m(v_2 - v_1)$, impulse is also said to be the change in momentum.

Energy: There are several different forms of energy which may exist in a system. These may be converted from one type to another but they can never be destroyed. Energy is measured in joules.

Potential energy (PE) is the energy which a body possesses by virtue of its position in relation to other bodies: $PE = mgh$, where h is the distance above some fixed datum and g is the acceleration due to gravity.

Kinetic energy (KE) is the energy a body possesses by virtue of its motion: $KE = \frac{1}{2}mv^2$.

Work (W) is a measure of the amount of energy produced when a force moves a body a given distance: $W = F \cdot x$.

Power (P) is the rate of doing work with respect to time and is measured in watts.

Moment of inertia (I): The moment of inertia is that property in a rotational system which may be considered equivalent to the mass in a translational system. It is defined about an axis xx as $I_{XX} = \Sigma \delta m x^2 = mk^2_{XX}$, where x is the perpendicular distance of an element of mass δm from the axis xx and k_{XX} is the radius of gyration about the axis xx. Table 3.1 gives some data on moments of inertia for standard shapes.

Angular velocity (ω) is the rate of change of angular distance (θ) with time:

$$= d\theta/dt = \dot{\theta}$$

Angular acceleration (α) is the rate of change of angular velocity (ω) with time:

$$= d\omega/dt \quad \text{or} \quad d^2\theta/dt^2 \quad \text{or} \quad \ddot{\theta}$$

Both angular velocity and accleration are related to linear motion by the equations $v = \omega x$ and $a = \alpha x$ (see Figure 3.2).

Torque (T) is the moment of force about the axis of rotation:

$$T = I_0 \alpha$$

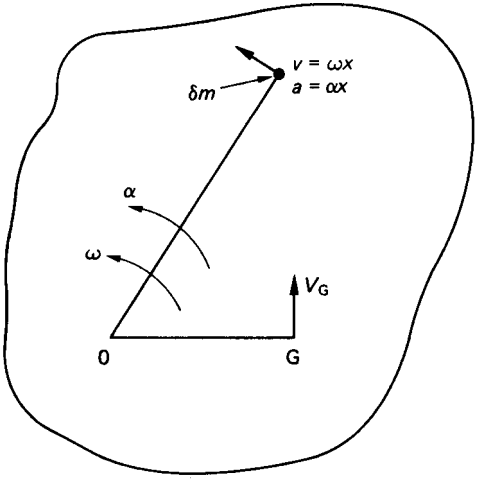


FIGURE 3.2

A torque may also be equal to a *couple,* which is two forces equal in magnitude acting some distance apart in opposite directions.

Parallel axis theorem: if I_{GG} is the moment of inertia of a body of mass m about its center of gravity, then the moment of inertia (I) about some other axis parallel to the original axis is given by $I = I_{GG} + mr^2$, where r is the perpendicular distance between the parallel axes.

Perpendicular axis theorem: If I_{XX}, I_{YY}, and I_{ZZ} represent the moments of inertia about three mutually perpendicular axes x, y, and z for a plane figure in the xy plane (see Figure 3.3) then $I_{ZZ} = I_{XX} + I_{YY}$.

Angular momentum (H_O) of a body about a point O is the moment of the linear momentum about that point and is ωI_{OO}. The angular momentum of a system remains constant unless acted on by an external torque.

Angular impulse is the product of torque by time, i.e. angular impulse $= Tt = I\alpha \cdot t = I(\omega_2 - \omega_1)$, the change in angular momentum.

Angular kinetic energy about an axis O is given by $\frac{1}{2}I_O\omega^2$.

Work done due to a torque is the product of torque by angular distance and is given by $T\theta$.

Power due to torque is the rate of angular work with respect to time and is given by $Td\theta/dt = T\omega$.

Friction: Whenever two surfaces, which remain in contact, move one relative to the other there is a force which acts tangentially to the surfaces so as to oppose motion. This is known as the force of friction. The magnitude of this force is μR, where R is the normal reaction and μ is a constant known as the coefficient of friction. The coefficient of friction depends on the nature of the surfaces in contact.

3.2 LINEAR AND ANGULAR MOTION IN TWO DIMENSIONS

Constant acceleration: If the accleration is integrated twice and the relevant initial conditions are used, then the following equations hold:

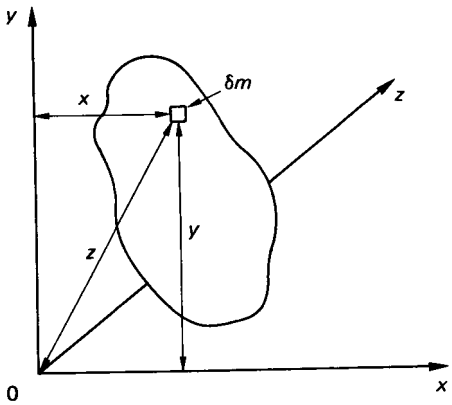


FIGURE 3.3

Linear motion	*Angular motion*
$x = v_1 t + \frac{1}{2}at^2$	$\theta = \omega_1 t + \frac{1}{2}\alpha t^2$
$v_2 = v_1 + at$	$\omega_2 = \omega_1 + \alpha t$
$v_2^2 = v_1^2 + 2ax$	$\omega_2^2 = \omega_1^2 + 2\alpha\theta$

Variable acceleration: If the acceleration is a function of time then the area under the acceleration time curve represents the change in velocity. If the acceleration is a function of displacement then the area under the acceleration distance curve represents half the difference of the square of the velocities (see Figure 3.4).

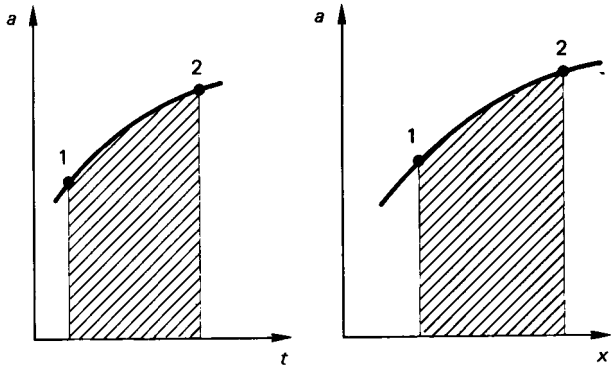
Curvilinear motion is when both linear and angular motions are present.

If a particle has a velocity v and an acceleration a then its motion may be described in the following ways:

1. *Cartesian components,* which represent the velocity and acceleration along two mutually perpendicular axes x and y (see Figure 3.5(a)):

$$v_X = v\cos\theta, \quad v_Y = v\sin\theta, \quad a_X = a\cos\phi$$

$$a_Y = a\sin\phi$$



$$a = \frac{dv}{dt} \text{ or } a\,dt = dv$$

$$\text{Area} = \int_1^2 a.dt$$

$$= v_2 - v_1$$

$$a = \frac{dx}{dt} \cdot \frac{dv}{dx}$$

$$a = v\frac{dv}{dx}$$

$$\text{or } a\,dx = v\,dv$$

$$\text{Area} = \int_1^2 a\,dx = \frac{v_2^2}{2} - \frac{v_1^2}{2}$$

FIGURE 3.4

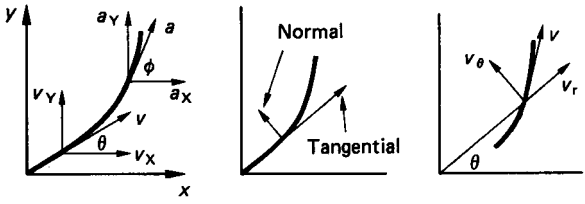


FIGURE 3.5

2. *Normal and tangential components:* see Figure 3.5(b):

$$v_t = v = r\dot{\theta} = r\omega, \quad v_n = 0$$

$$a_t = r\dot{\theta} + \dot{r}\dot{\theta} = r\alpha + \dot{r}\omega$$

$$a_n = v\dot{\theta} = r\omega^2$$

3. *Polar coordinates:* see Figure 3.5(c):

$$v_t = \dot{r}, \quad v_\theta = r\dot{\theta}$$

$$a_t = \ddot{r} - r\dot{\theta}^2, \quad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

3.3 CIRCULAR MOTION

Circular motion is a special case of curvilinear motion in which the radius of rotation remains constant. In this case there is an acceleration towards the center of $\omega^2 r$. This gives rise to a force towards the center known as the *centripetal force*. This force is reacted to by what is called the *centrifugal reaction*.

Velocity and acceleration in mechanisms: A simple approach to determine the velocity and acceleration of a mechanism at a point in time is to draw velocity and acceleration vector diagrams.

Velocities: If in a rigid link AB of length l the end A is moving with a different velocity to the end B, then the velocity of A relative to B is in a direction perpendicular to AB (see Figure 3.6).

When a block slides on a rotating link the velocity is made up of two components, one being the velocity of the block relative to the link and the other the velocity of the link.

Accelerations: If the link has an angular acceleration α then there will be two components of acceleration in the diagram, a tangential component αl and a centripetal component of magnitude $\omega^2 l$ acting towards A.

When a block slides on a rotating link the total acceleration is composed of four parts: first, the centripetal acceleration towards O of magnitude $\omega^2 l$; second, the tangential acceleration αl; third, the acceleration of the block relative to the link; fourth, a tangential acceleration of magnitude $2v\omega$ known as Coriolis acceleration.

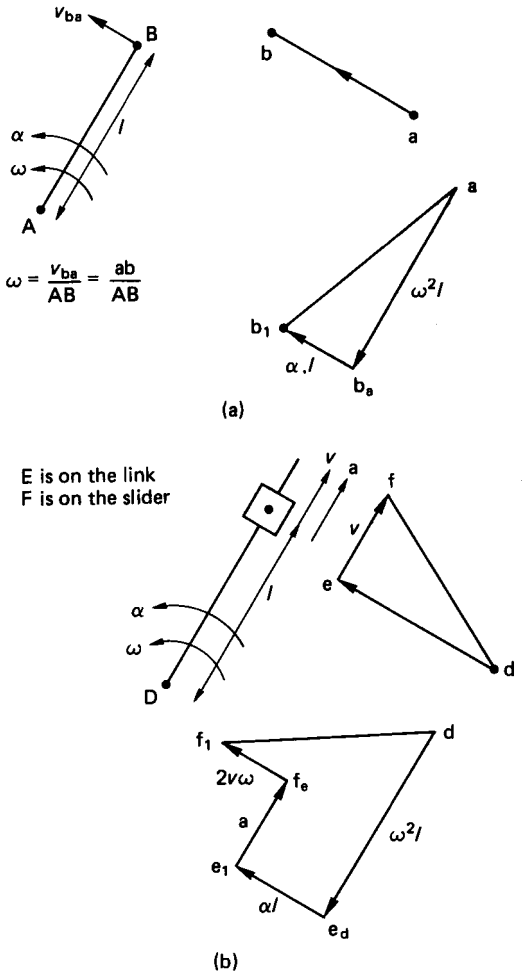


FIGURE 3.6

The direction of Coriolis acceleration is determined by rotating the sliding velocity vector through 90° in the direction of the link angular velocity ω.

3.4 LINEAR AND ANGULAR MOTION IN THREE DIMENSIONS

Motion of a Particle in a Moving Coordinate System

xyz is a moving coordinate system, with its origin at *O*, which has a position vector $\mathbf{R}$, a translational velocity vector $\dot{\mathbf{R}}$ and an angular velocity vector $\boldsymbol{\omega}$ relative to a

fixed coordinate system XYZ, origin at O'. Then the motion of a point P whose position vector relative to O is $\mathbf{P}$ and relative to O' is $\mathbf{r}$ is given by the following equations (see Figure 3.7):

$$\dot{\mathbf{r}} = \dot{\mathbf{R}} + \dot{\boldsymbol{\rho}}_r + \boldsymbol{\omega} \times \boldsymbol{\rho}$$

where $\boldsymbol{\rho}_r$ is the velocity of the point P relative to the moving system xyz and $\boldsymbol{\omega} \times \boldsymbol{\rho}$ is the vector product of $\boldsymbol{\omega}$ and $\boldsymbol{\rho}$:

$$\ddot{\mathbf{r}} = \ddot{\mathbf{R}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) + 2\boldsymbol{\omega} \times \dot{\boldsymbol{\rho}}_r + \ddot{\boldsymbol{\rho}}_t$$

where $\boldsymbol{\rho}_r$ is the acceleration of the point P relative to the moving system. Thus $\dot{\boldsymbol{r}}$ is the sum of:

1. The relative velocity $\ddot{\boldsymbol{\rho}}_r$
2. The absolute velocity $\mathbf{R}$ of the moving origin O
3. The velocity $\boldsymbol{\omega} \times \boldsymbol{\rho}$ due to the angular velocity of the moving axes xyz

and $\ddot{\mathbf{r}}$ is the sum of:

1. The relative acceleration $\ddot{\boldsymbol{\rho}}_r$
2. The absolute acceleration $\ddot{\mathbf{R}}$ of the moving origin O
3. The tangential acceleration $\dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}$ due to the angular acceleration of the moving axes xyz
4. The centripetal acceleration $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})$ due to the angular velocity of the moving axes xyz
5. Coriolis component acceleration $2\boldsymbol{\omega} \times \boldsymbol{\rho}_r$ due to the interaction of coordinate angular velocity and relative velocity

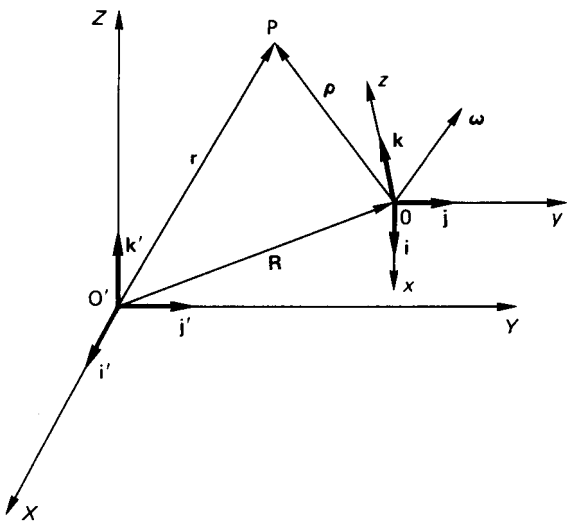


FIGURE 3.7

In all the vector notation a right-handed set of coordinate axes and the right-hand screw rule is used.

Gyroscopic Effects

Consider a rotor which spins about its geometric axis (see Figure 3.8) with an angular velocity ω. Then two forces F acting on the axle to form a torque T, whose vector is along the x-axis, will produce a rotation about the y axis. This is known as precession, and it has an angular velocity Ω. It is also the case that if the rotor is precessed then a torque T will be produced, where T is given by $T = I_{XX}\omega\Omega$. When this is observed it is the effect of gyroscopic reaction torque that is seen, which is in the opposite direction to the gyroscopic torque (Scarborough 1958).

3.5 BALANCING

In any rotational or reciprocating machine where accelerations are present, unbalanced forces can lead to high stresses and vibrations. The principle of balancing is such that by the addition of extra masses to the system the out-of-balance forces may be reduced or eliminated.

3.6 BALANCING OF ROTATING MASSES

Single Out-of-Balance Mass

One mass (m) at a distance r from the center of rotation and rotating at a constant angular velocity ω produces a force $m\omega^2 r$. This can be balanced by a mass M placed diametrically opposite at a distance R, such that $MR = mr$.

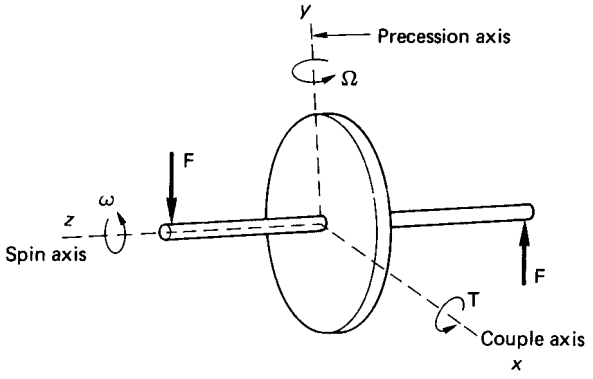


FIGURE 3.8

Several Out-of-Balance Masses in One Transverse Plane

If a number of masses (m_1, m_2, . . .) are at radii (r_1, r_2, . . .) and angles (θ_1, θ_2, . . .) (see Figure 3.9), then the balancing mass M must be placed at a radius R such that MR is the vector sum of all the mr terms.

Masses in Different Transverse Planes

If the balancing mass in the case of a single out-of-balance mass were placed in a different plane then the centrifugal force would be balanced. This is known as *static balancing*. However, the moment of the balancing mass about the original plane would lead to what is known as *dynamic unbalance.*

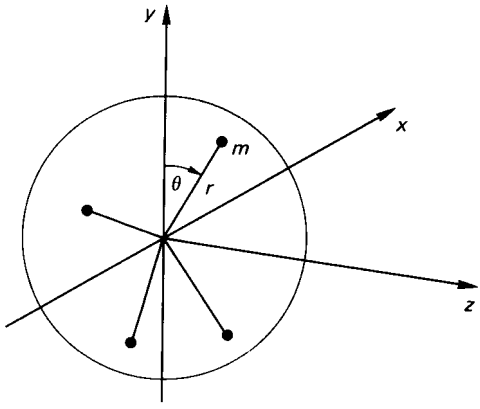
To overcome this, the vector sum of all the moments about the reference plane must also be zero. In general, this requires two masses placed in convenient planes (see Figure 3.10).

Balancing of Reciprocating Masses in Single-Cylinder Machines

The acceleration of a piston as shown in Figure 3.11 may be represented by the equation (Wilson 1959)

$$\ddot{x} = -\omega^2 r[\cos\theta + (1/n)\cos 2\theta + (1/4n)(\cos 2\theta - \cos 4\theta) + \ldots .*$$

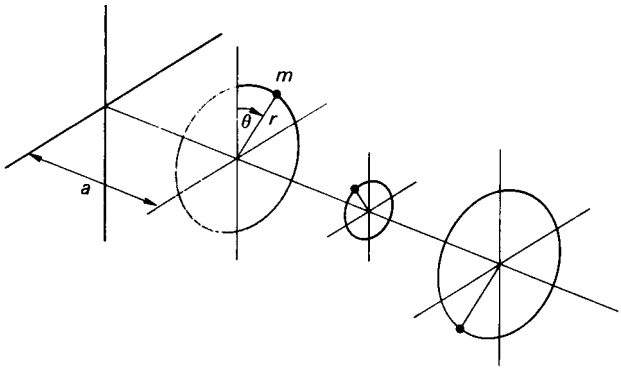
where $n = 1/r$. If n is large, then the equation may be simplified and the force given by



$\Sigma F_X = \Sigma m\omega^2 r \sin\theta = 0$
$\Sigma F_Y = \Sigma m\omega^2 r \cos\theta = 0$

FIGURE 3.9

*This equation forms an infinite series in which higher terms are small, and they may be ignored for practical situations.



$\Sigma F_X = \Sigma m\omega^2 r \sin\theta = 0$ and $\Sigma F_Y = \Sigma m\omega^2 r \cos\theta = 0$
as in the previous case, also
$\Sigma M_X = \Sigma m\omega^2 r \sin\theta \,.\, a = 0$
$\Sigma M_Y = \Sigma m\omega^2 r \cos\theta \,.\, a = 0$

FIGURE 3.10

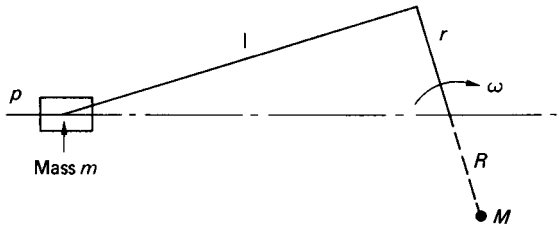


FIGURE 3.11

$$F = m\ddot{x} \simeq -m\omega^2 r[\cos\theta + (1/n)\cos 2\theta]$$

The term $m\omega^2 r \cos\theta$ is known as the *primary force* and $(1/n)m\omega^2 r \cos 2\theta$ as the *secondary force.* Partial primary balance is achieved in a single-cylinder machine by an extra mass M at a radius R rotating at the crankshaft speed. Partial secondary balance could be achieved by a mass rotating at 2ω. As this is not practical this is not attempted. When partial primary balance is attempted a transverse component $M\omega^2 R \sin\theta$ is introduced. The values of M and R are chosen to produce a compromise between the reciprocating and the transverse components.

Balancing of Reciprocating Masses in Multicylinder Machines

When considering multicylinder machines, account must be taken of the force produced by each cylinder and the moment of that force about some datum. The conditions for primary balance are

$$F = \Sigma m\omega^2 r \cos\theta = 0, \quad M = \Sigma m\omega^2 r \cos\theta \cdot a = 0$$

where a is the distance of the reciprocating mass m from the datum plane.

MECHANICAL ENGINEERING PRINCIPLES

In general, the cranks in multicylinder engines are arranged to assist primary balance. If primary balance is not complete, then extra masses may be added to the crankshaft but these will introduce an unbalanced transverse component. The conditions for secondary balance are

$$F = \Sigma m\omega^2(r/n)\cos 2\theta = \Sigma m(2\omega)^2(r/4n)\cos 2\theta = 0$$

and

$$M = \Sigma m(2\omega)^2(r/4n)\cos 2\theta \cdot a = 0$$

The addition of extra masses to give secondary balance is not attempted in practical situations.

PART 4

VIBRATIONS

Robert Paine

Christopher Beards

3.7 SINGLE-DEGREE-OF-FREEDOM SYSTEMS

The term *degrees of freedom* in an elastic vibrating system is the number of parameters required to define the configuration of the system. To analyze a vibrating system, a mathematical model is constructed, which consists of springs and masses for linear vibrations. The type of analysis then used depends on the complexity of the model.

Rayleigh's method: Rayleigh showed that if a reasonable deflection curve is assumed for a vibrating system, then by considering the kinetic and potential energies* an estimate to the first natural frequency could be found. If an inaccurate curve is used then the system is subject to constraints to vibrate it in this unreal form, and this implies extra stiffness such that the natural frequency found will always be high. If the exact deflection curve is used then the natural frequency will be exact.

Transverse Vibration of Beams

Consider a beam of length (l), weight per unit length (w), modulus (E), and moment of inertia (I). Then its equation of motion is given by

*Consider the equation of motion for an undamped system (Figure 3.13):

$$m \cdot \frac{d^2x}{dt^2} + kx = 0 \tag{3.4}$$

but

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{dx}{dt} \cdot \frac{d}{dx}\left(\frac{dx}{dt}\right) = \frac{1}{2} \cdot \frac{d}{dx}\left(\frac{dx}{dt}\right)^2$$

Therefore, equation (3.4) becomes

$$\frac{d}{dx}\left[\frac{1}{2}m\left(\frac{dx}{dt}\right)^2\right] + kx = 0$$

Integrating gives

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}kx^2 = \text{Constant}$$

the term $\frac{1}{2}m(dx/dt)^2$ represents the kinetic energy and $\frac{1}{2}kx^2$ the potential energy.

MECHANICAL ENGINEERING PRINCIPLES

$$EI\frac{d^4y}{dx^4} - w\omega^2 y/g = 0$$

where ω is the natural frequency. The general solution of this equation is given by

$$y = A\cos\beta x + B\sin\beta x + C\cosh\beta x + D\sinh\beta x$$

where $\beta^4 = w\omega^2/gEI$.

The four constants of integration A, B, C, and D are determined by four independent end conditions. In the solution trigonometrical identities are formed in β which may be solved graphically, and each solution corresponds to a natural frequency of vibration. Table 3.3 shows the solutions and frequencies for standard beams (Young and Feglar 1949).

Dunkerley's empirical method is used for beams with multiple loads. In this method the natural frequency (f_1) is found due to just one of the loads, the rest being ignored. This is repeated for each load in turn and then the natural frequency of vibration of the beam due to its weight alone is found (f_0). Then the natural frequency of vibration of the complete system (f) is given by

$$\frac{1}{f^2} = \frac{1}{f_0^2} + \frac{1}{f_1^2} + \frac{1}{f_2^2} + \frac{1}{f_3^2} + \cdots \frac{1}{f_n^2}$$

(see Cole 1950 for a more detailed explanation).

Whirling of shafts: If the speed of a shaft or rotor is slowly increased from rest there will be a speed where the deflection increases suddenly. This phenomenon is known as whirling. Consider a shaft with a rotor of mass m such that the center of gravity is eccentric by an amount e. If the shaft now rotates at an angular velocity ω then the shaft will deflect by an amount y due to the centrifugal reaction (see Figure 3.12). Then

$$m\omega^2(y + e) = ky$$

where k is the stiffness of the shaft. Therefore,

$$y = \frac{e}{(k/m\omega^2 - 1)}$$

When $(k/m\omega^2) = 1$, y is then infinite and the shaft is said to be at its critical whirling speed ω_c. At any other angular velocity ω the deflection y is given by

$$y = \left(\frac{\omega^2}{\omega_c^2 - \omega^2}\right) \cdot e$$

When $\omega < w_c$, y is the same sign as e and as ω increases towards ω_c the deflection theoretically approaches infinity. When $\omega > \omega_c$, y is opposite in sign to e and will eventually tend to $-e$. This is a desirable running condition with the center of gravity of the rotor mass on the static deflection curve. Care must be taken not to increase ω too high, as ω might start to approach one of the higher modes of vibration (Thompson 1983).

Torsional vibrations: The following section deals with transverse vibrating systems with displacements x and masses m. The same equations may be used for

TABLE 3.3

End conditions	Trigometric equation	Solutions		
		$\beta_1 l$	$\beta_2 l$	$\beta_3 l$
$x = 0,\ y = 0,\ y' = 0$ $x = 1,\ y = 0,\ y' = 0$	$\cos \beta l \cdot \cosh \beta l = 1$	4.730	7.853	10.966
$x = 0,\ y = 0,\ y' = 0$ $x = 1,\ y'' = 0,\ y''' = 0$	$\cos \beta l \cdot \cosh \beta l = -1$	1.875	4.694	7.855
$x = 0,\ y = 0,\ y'' = 0$ $x = 1,\ y = 0,\ y'' = 0$	$\sin \beta l = 0$	3.142	6.283	9.425
$x = 0,\ y = 0,\ y' = 0$ $x = 1,\ y = 0,\ y'' = 0$	$\tan \beta l = \tanh \beta l$	3.927	7.069	10.210

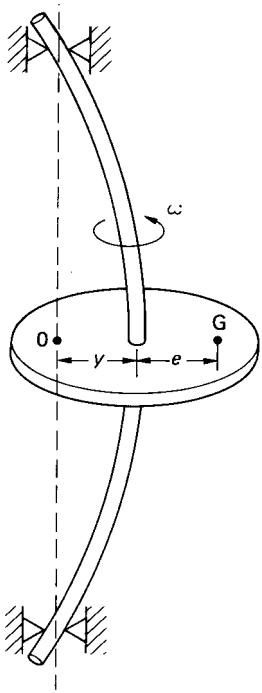


FIGURE 3.12

torsional vibrating systems by replacing x by θ the angular displacement and m by I, the moment of inertia.

Undamped Free Vibrations

The equation of motion is given by $m\ddot{x} + kx = 0$ or $\ddot{x} + \omega_n^2 x = 0$, where m is the mass, k the stiffness and $\omega_n^2 = k/m$, which is the natural frequency of vibration of the system (see Figure 3.13). The solution to this equation is given by

$$x = A \sin(\omega_n t + \alpha)$$

where A and α are constants which depend on the initial conditions. This motion is said to be *simple harmonic* with a time period $T = 2\pi/\omega_n$.

Damped Free Vibrations

The equation of motion is given by $m\ddot{x} + c\dot{x} + kx = 0$ (see Figure 3.14), where c is the viscous damping coefficient, or $\ddot{x} + (c/m)\dot{x} + \omega_n^2 x = 0$. The solution to this equation and the resulting motion depends on the amount of damping. If $c > 2m\omega_n$ the system is said to be overdamped. It will respond to a disturbance by slowly

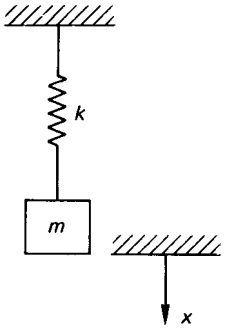


FIGURE 3.13

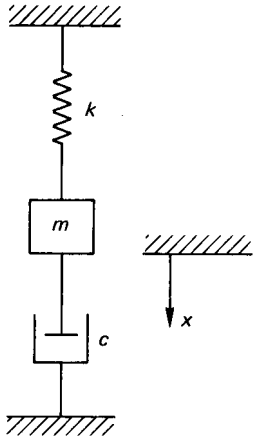


FIGURE 3.14

returning to its equilibrium position. The time taken to return to this position depends on the degree of damping (see Figure 3.15(c)). If $c = 2m\omega_n$ the system is said to be critically damped. In this case it will respond to a disturbance by returning to its equilibrium position in the shortest possible time. In this case (see Figure 3.15(b))

$$x = e^{-(c/2m)t(A+Bt)}$$

where A and B are constants. If $c < 2m\omega_n$, the system has a transient oscillatory motion given by

$$x = e^{-(c/2m)t}[C \sin(\omega_n^2 - c^2/4m^2)^{1/2}t + D(\cos \omega_n^2 - c^2/4m^2)^{1/2}t]$$

where C and D are constants. The period

$$T = \frac{2\pi}{(\omega_n^2 - c^2/4m^2)^{1/2}}$$

(see Figure 3.15(a)).

Logarithmic Decrement. A way to determine the amount of damping in a system is to measure the rate of decay of successive oscillations. This is expressed by a term called the *logarithmic decrement* (δ), which is defined as the natural logarithm of the ratio of any two successive amplitudes (see Figure 3.16):

$$\delta = \log_e(x_1/x_2)$$

where x is given by

$$x = e^{cT/m} \sin\left[\left(\omega^2 - \frac{c^2}{4m^2}\right)^{1/2} + \phi\right]$$

Therefore,

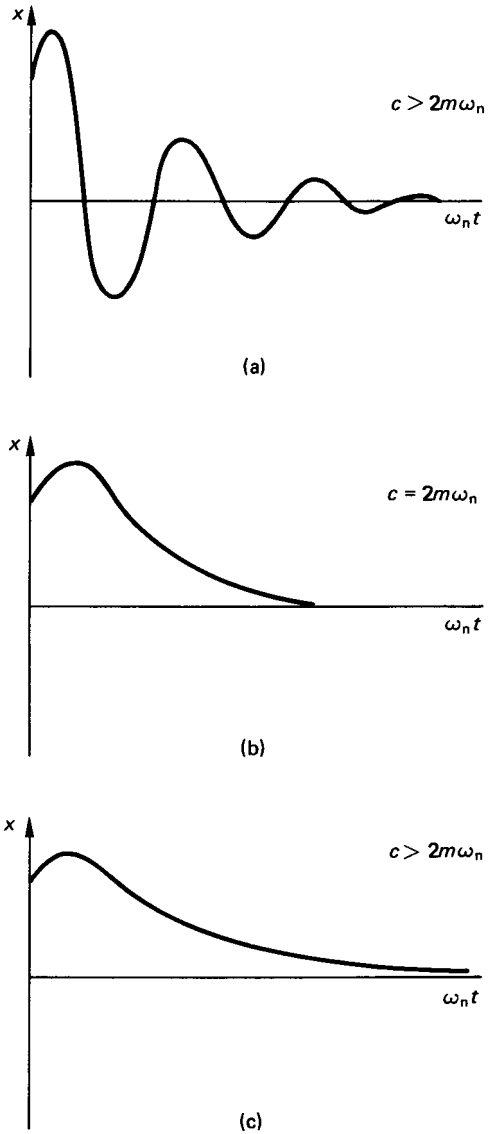


FIGURE 3.15

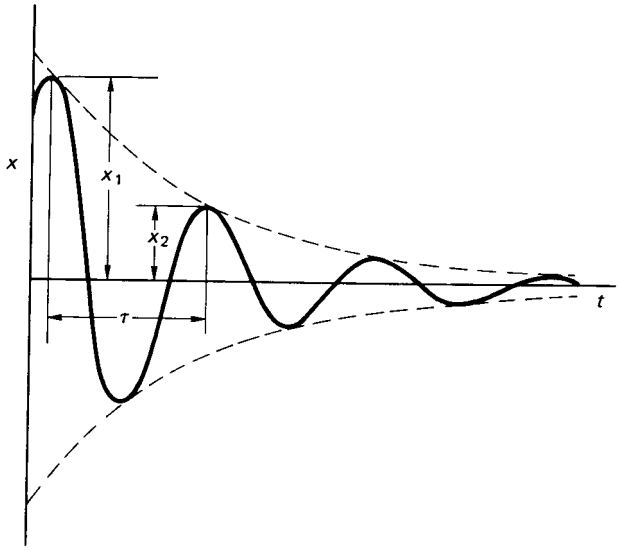


FIGURE 3.16

$$\delta = \log_e(e^{-cT/2m}/e^{-c(T+\tau)/2m})$$
$$= c\tau/2m$$

where τ is the period of damped oscillation.

If the amount of damping present is small compared to the critical damping, τ approximates to $2\pi/\omega$, and then

$$\delta = c\pi/m\omega_n$$

Forced Undamped Vibrations

The equation of motion is given by (see Figure 3.17)

$$m\ddot{x} + kx = F_0 \sin \omega t$$

or

$$\ddot{x} + \omega_n^2 = (F_0/m) \sin \omega t$$

The solution to this equation is

$$x = C \sin \omega_n t + D \cos \omega_n t + F_0 \cos \omega t[m(\omega_n^2 - \omega^2)]$$

where ω is the frequency of the forced vibration. The first two terms of the solution are the transient terms, which die out, leaving an oscillation at the forcing frequency of amplitude

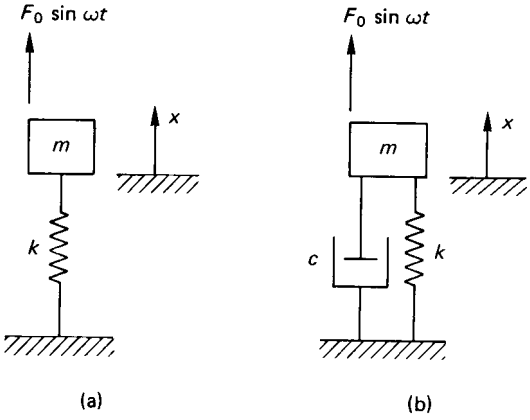


FIGURE 3.17

$$F_0/[m(\omega_n^2 - \omega^2)]$$

or

$$\frac{F_0}{k}\left(\frac{\omega_n^2}{\omega_n^2 - \omega^2}\right)$$

The term $\omega_n^2/(\omega_n^2 - \omega^2)$ is known as the dynamic magnifier, and it gives the ratio of the amplitude of the vibration to the static deflection under the load F_0. When $\omega = \omega_n$ the amplitude becomes infinite and resonance is said to occur.

Forced Damped Vibrations

The equation of motion is given by (see Figure 3.17(b))

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega t$$

or

$$\ddot{x} + (c/m)\dot{x} + \omega_n^2 = (F_0/m) \sin \omega t$$

The solution to this equation is in two parts: a transient part as in the undamped case which dies away, leaving a sustained vibration at the forcing frequency given by

$$x = \frac{F_0}{m} \frac{1}{[(\omega_n^2 - \omega^2)^2 + (c(\omega/m)^2]^{1/2}} \sin(\omega t - \alpha)$$

The term

$$\frac{\omega_n^2}{[(\omega_n^2 - \omega^2)^2 + (c\omega/m)]^2]^{1/2}}$$

is called the dynamic magnifier. Resonance occurs when $\omega \simeq \omega_n$. As the damping

is increased the value of ω for which resonance occurs is reduced. There is also a phase shift as ω increases tending to a maximum of π radians. It can be seen in Figure 3.18(a) that when the forcing frequency is high compared to the natural frequency the amplitude of vibration is minimized.

Forced Damped Vibrations Due to Reciprocating or Rotating Unbalance

Figure 3.19 shows two elastically mounted systems, (a) with the excitation supplied by the reciprocating motion of a piston, and (b) by the rotation of an unbalanced rotor. In each case the equation of motion is given by

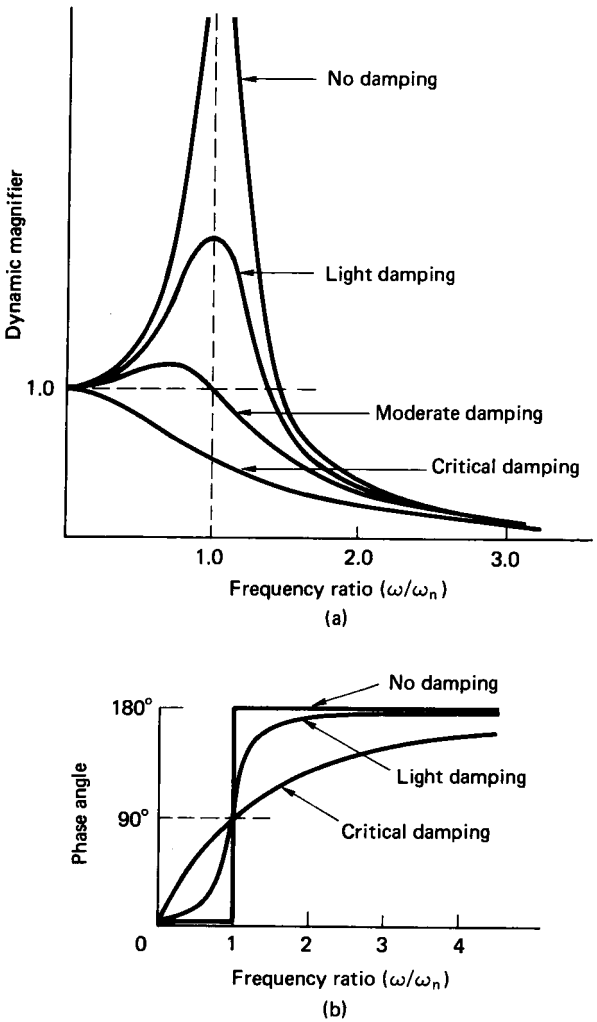


FIGURE 3.18

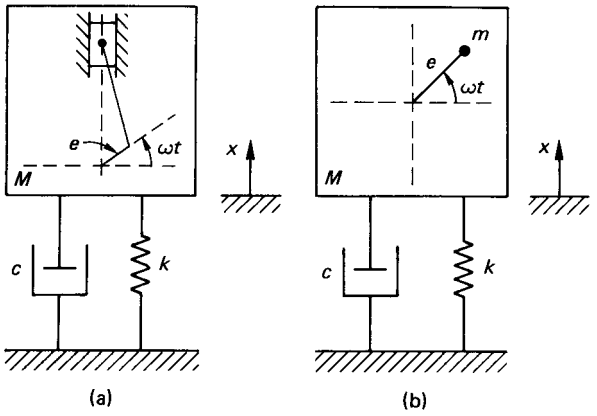


FIGURE 3.19

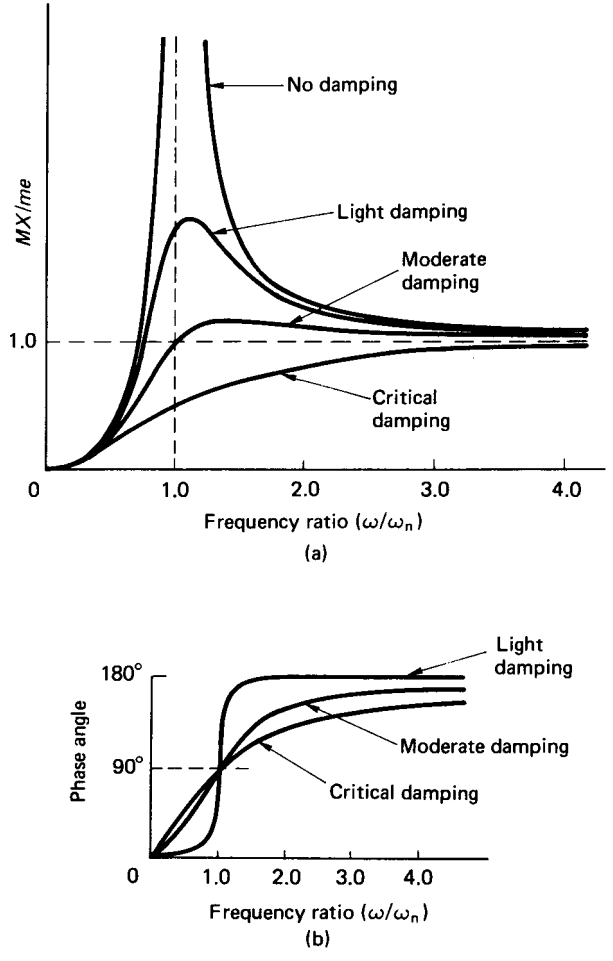


FIGURE 3.20

$$(M - m)\ddot{x} + c\dot{x} + kx = (me\omega^2) \sin \omega t$$

The solution of this equation is a sinusoid whose amplitude, X, is given by

$$X = \frac{me\omega^2}{\sqrt{[(K - M\omega^2)^2 + (c\omega)^2]}}$$

In representing this information graphically it is convenient to plot MX/me against ω/ω_n for various levels of damping (see Figure 3.20(a)). From this figure it can be seen that for small values of ω the displacement is small, and as ω is increased the displacement reaches a maximum when ω is slightly greater than ω_n. As ω is further increased the displacement tends to a constant value such that the center of gravity of the total mass M remains stationary. Figure 3.20(b) shows how the phase angle varies with frequency.

Forced Damped Vibration. If a system as shown in Figure 3.21 has a sinusoidal displacement applied to its base of amplitude, y, then the equation of motion becomes

$$m\ddot{x} + c\dot{x} + kx = ky + c\dot{y}$$

The solution of this equation yields

$$\frac{x}{y} = \sqrt{\left[\frac{k^2 + (c\omega)^2}{(k - m\omega^2)^2 + (c\omega)^2}\right]}$$

where x is the amplitude of motion of the system.

When this information is plotted as in Figure 3.22, it can be seen that for very small values of ω the output amplitude X is equal to the input amplitude Y. As ω is increased towards ω_n the output reaches a maximum. When $\omega = \sqrt{2}\ \omega_n$ the curves intersect and the effect of damping is reversed.

The curves in Figure 3.22 may also be used to determine the amount of sinusoidal force transmitted through the springs and dampers to the supports, i.e., the axis (X/Y) may be replaced by (F_t/F_0) where F_0 is the amplitude of applied force and F_t is the amplitude of force transmitted.

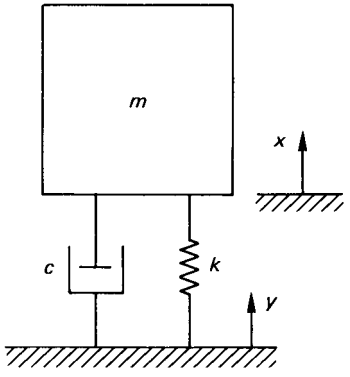


FIGURE 3.21

MECHANICAL ENGINEERING PRINCIPLES

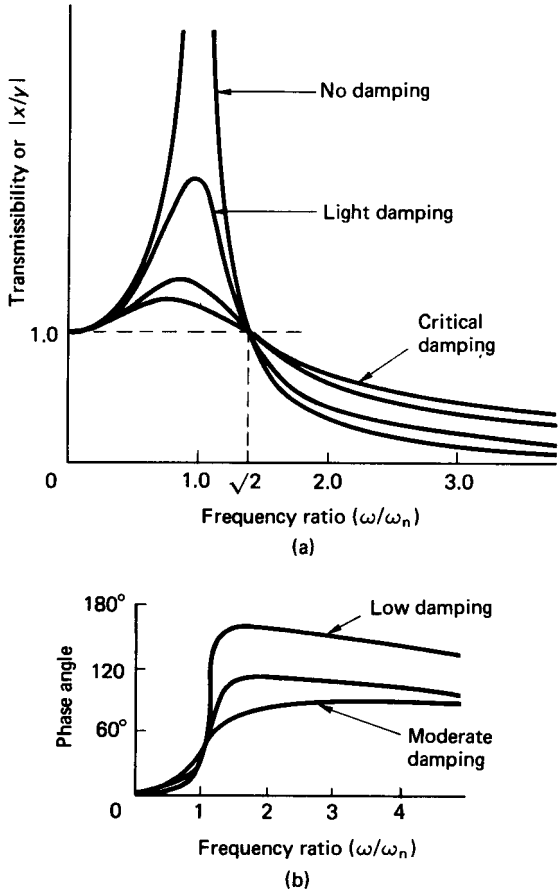


FIGURE 3.22

3.8 MULTI-DEGREE-OF-FREEDOM SYSTEMS

Normal Mode Vibration

The fundamental techniques used in modeling multi-degree-of-freedom systems may be demonstrated by considering a simple two-degree-of-freedom system as shown in Figure 3.23. The equations of motion for this system are given by

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0$$

$$m_2\ddot{x}_2 + (k_3 + k_2)x_2 - k_2x_1 = 0$$

or in matrix form:

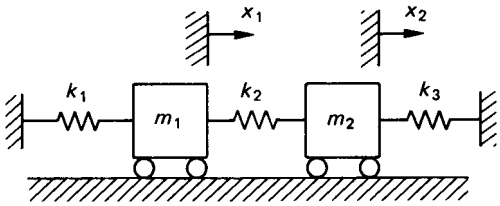


FIGURE 3.23

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & (k_3 + k_2) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Assuming the motion of every point in the system to be harmonic, then the solutions will take the form

$$x_1 = A_1 \sin \omega t$$

$$x_2 = A_2 \sin \omega t$$

where A_1 and A_2 are the amplitudes of the respective displacements. By substituting the values of x_1, x_2, $\ddot{x}_1$, and $\ddot{x}_2$ into the original equations the values of the natural frequencies of vibration may be found along with the appropriate mode shapes. This is a slow and tedious process, especially for systems with large numbers of degrees of freedom, and is best performed by a computer program.

The Holtzer Method

When only one degree of freedom is associated with each mass in a multimass system, then a solution can be found by proceeding numerically from one end of the system to the other. If the system is being forced to vibrate at a particular frequency, then there must be a specific external force to produce this situation. A frequency and a unit deflection is assumed at the first mass and from this the inertia and spring forces are calculated at the second mass. This process is repeated until the force at the final mass is found. If this force is zero, then the assumed frequency is a natural frequency. Computer analysis is most suitable for solving problems of this type.

Consider several springs and masses as shown in Figure 3.24. Then with a unit deflection at the mass m_1 and an assumed frequency ω there will be an inertia

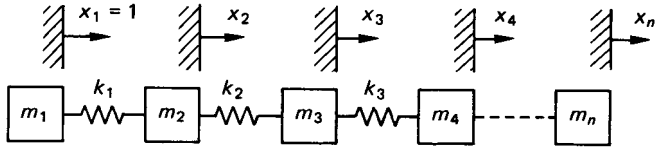


FIGURE 3.24

force of $m_1\omega^2$ acting on the spring with stiffness k_1. This causes a deflection of $m_1\omega^2/k_1$, but if m_2 has moved a distance x_2 then $m_1\omega^2/k_1 = 1 - x_2$ or $x_2 = 1 - m_1\omega^2/k_1$. The inertia force acting due to m_2 is $m_2\omega^2x_2$, thus giving the total force acting on the spring of stiffness k_2 as $[m_1\omega^2 + m_2\omega^2x_2]/k_2$. Hence the displacement at x_3 can be found and the procedure repeated. The external force acting on the final mass is then given by

$$\sum_{i=1}^{n} m_i\omega^2 x_i.$$

If this force is zero, then the assumed frequency is a natural one.

3.9 RANDOM VIBRATIONS

Introduction

If the vibration response parameters of a dynamic system are accurately known as functions of time, the vibration is said to be *deterministic*. However, in many systems and processes responses cannot be accurately predicted; these are called *random processes*. Examples of a random process are turbulence, fatigue, the meshing of imperfect gears, surface irregularities, the motion of a car running along a rough road, and building vibration excited by an earthquake (Figure 3.25).

A collection of sample functions $x_1(t)$, $x_2(t)$, $x_3(t)$, . . . , $x_n(t)$ which make up the random process $x(t)$ is called an *ensemble* (Figure 3.26). These functions may comprise, for example, records of pressure fluctuations or vibration levels, taken under the same conditions but at different times.

Any quantity which cannot be precisely predicted is nondeterministic and is known as a *random variable* or a *probabilistic quantity*. That is, if a series of tests are conducted to find the value of a particular parameter, x, and that value is found to vary in an unpredictable way that is not a function of any other parameter, then x is a random variable.

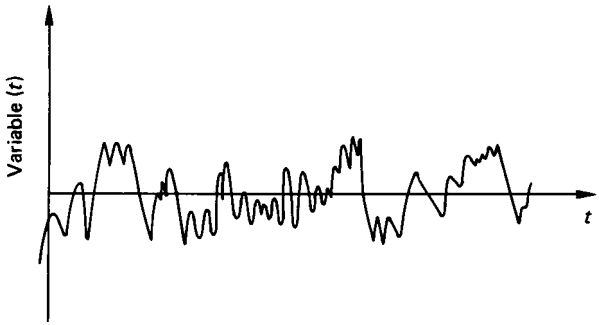


FIGURE 3.25 Example random process variable as $f(t)$.

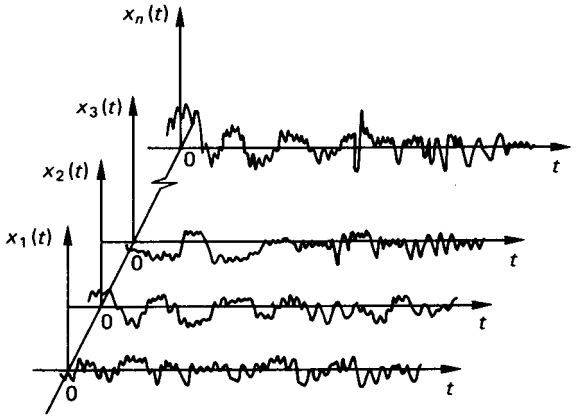


FIGURE 3.26 Ensemble of a random process.

Probability Distribution

If n experimental values of a variable x are $x_1, x_2, x_3, \ldots . x_n$, the probability that the value of x will be less than x' is n'/n, where n' is the number of x values which are less than or equal to x'. That is,

$$\text{Prob}(x \leqslant x') = n'/n$$

When n approaches ∞ this expression is the probability distribution function of x, denoted by $P(x)$, so that

$$P(x) = \underset{n\to\infty}{\text{Lt}} \; (n'/n)$$

The typical variation of $P(x)$ with x is shown in Figure 3.27. Since $x(t)$ denotes a physical quantity,

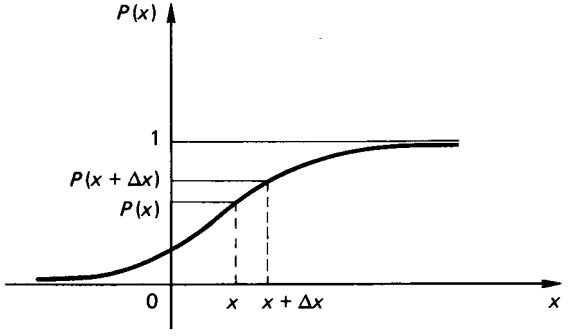


FIGURE 3.27 Probability distribution function as $f(x)$.

MECHANICAL ENGINEERING PRINCIPLES

$$\text{Prob}(x < -\infty) = 0, \text{ and } \text{Prob}(x < +\infty) = 1$$

The *probability density function* is the derivative of $P(x)$ with respect to x and this is denoted by $p(x)$. That is,

$$p(x) = \frac{dP(x)}{dx}$$

$$= \underset{\Delta x \to 0}{\text{Lt}} \left[\frac{P(x + \Delta x) - P(x)}{\Delta x} \right]$$

where $P(x + \Delta x) - P(x)$ is the probability that the value of $x(t)$ will lie between x and $x + \Delta x$ (Figure 3.27). Now

$$p(x) = \frac{dP(x)}{dx}$$

so that

$$P(x) = \int_{-\infty}^{x} p(x)dx$$

Hence,

$$P(\infty) = \int_{-\infty}^{\infty} p(x)dx = 1$$

so that the area under the probability density function curve is unity.

A random process is *stationary* if the joint probability density

$$p(x)(t_1), x(t_2), x(t_3), \ldots)$$

depends only on the time differences $t_2 - t_1$, $t_3 - t_2$, and so on, and not on the actual time instants. That is, the ensemble will look just the same if the time origin is changed. A random process is *ergodic* if every sample function is typical of the entire group.

The expected value of $f(x)$, which is written $E[f(x)]$ or $\overline{f(x)}$ is

$$E[f(x)] = \overline{f(x)} = \int_{-\infty}^{\infty} f(x)p(x)dx$$

so that the expected value of a stationary random process $x(t)$ is

$$E[x(t_1)] = E[x(t_1 + t)]$$

for any value of t.

If $f(x) = x$, the expected value or *mean value* of x, $E[x]$ or $\bar{x}$, is

$$E[x] = \bar{x} = \int_{-\infty}^{\infty} xp(x)dx$$

In addition, if $f(x) = x^2$, the *mean square value* of x, $\bar{x}^2$ is

$$E[x^2] = \bar{x}^2 = \int_{-\infty}^{\infty} x^2 \cdot p(x)dx$$

The *variance* of x, σ^2 is the mean square value of x about the mean, that is,

$$\sigma^2 = E[(x - \bar{x})^2] = \int_{-\infty}^{\infty} (x - \bar{x})^2 p(x)dx = \overline{(x^2)} - (\bar{x})^2$$

σ is the *standard deviation* of x, hence

$$\text{Variance} = (\text{Standard deviation})^2$$
$$= \{\text{Mean square} - (\text{Mean})^2\}$$

If two or more random variables x_1 and x_2 represent a random process at two different instants of time, then

$$E[f(x_1,x_2)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1,x_2)p(x_1,x_2)dx_1dx_2$$

and if t_1 and t_2 are the two instants of time,

$$E[x(t_1),x(t_2)] = R(t_1,t_2)$$

which is the *autocorrelation function* for the random process (Figure 3.28).

For random processes which are stationary,

$$E[x(t_1),x(t_2)] = R(t_1,t_2) = R(t_2 - t_1) = R(\tau), \text{ say,}$$

since the average depends only on time differences. If the process is also ergodic, then

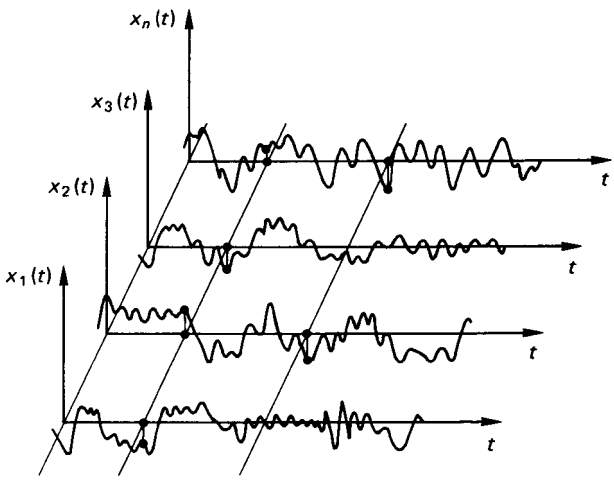


FIGURE 3.28 Random processes.

$$R(\tau) = \underset{T\to\infty}{\text{Lt}} \frac{1}{2\pi} \int_{-T}^{T} x(t)x(t+\tau)dt$$

It is worth noting that

$$R(0) = E[x(t)^2] = \underset{T\to\infty}{\text{Lt}} \frac{1}{2T} \int_{-T}^{T} x^2(t)dt$$

which is the average power in a sample function.

Random Processes

The most important random process is the *Gaussian* or *normal random process*. This is because a wide range of physically observed random waveforms can be represented as Gaussian processes, and the process has mathematical features which make analysis relatively straightforward.

The probability density function of a Gaussian process $x(t)$ is

$$p(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-1/2[(x-\bar{x})/\sigma]^2]}$$

where σ is the standard deviation of x and $\bar{x}$ is the mean value of x. The values of σ and $\bar{x}$ may vary with time for a non-stationary process but are independent of time if the process is stationary.

One of the most important features of the Gaussian process is that the response of a linear system to this form of excitation is usually another (but still Gaussian) random process. The only changes are that the magnitude and standard deviation of the response may differ from those of the excitation.

A Gaussian probability density function is shown in Figure 3.29. It can be seen to be symmetric about the mean value $\bar{x}$, and the standard deviation σ controls the spread.

The probability that $x(t)$ lies between $-\lambda\sigma$ and $+\lambda\sigma$, where λ is a positive number, can be found since, if $\bar{x} = 0$,

$$\text{Prob}\{-\lambda\sigma \le x(t) \le +\lambda\sigma\} = \int_{-\lambda\sigma}^{+\lambda\sigma} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-1/2(x^2/\sigma^2)}dx$$

Figure 3.30 shows the Gaussian probability density function with zero mean. This integral has been calculated for a range of values of λ and the results are given in Table 3.4. The probability that $x(t)$ lies outside the range $-\lambda\sigma$ to $+\lambda\sigma$ is 1 minus the value of the above integral. This probability is also given in Table 3.4.

Spectral Density

The spectral density $S(\omega)$ of a stationary random process is the Fourier transform of the autocorrelation function $R(\tau)$, and is given by

$$S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\tau)e^{-i\omega\tau}d\tau$$

The inverse, which also holds true, is

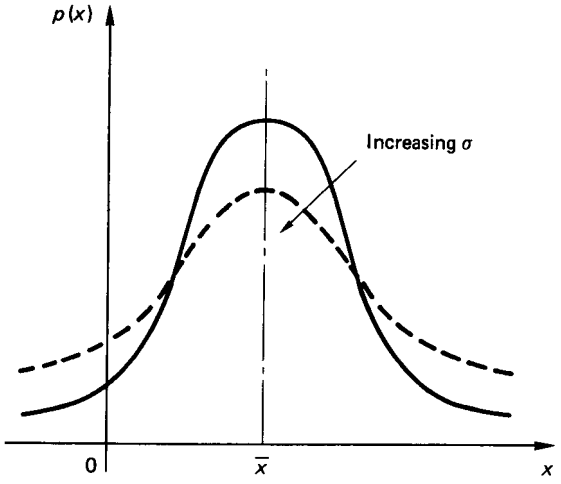


FIGURE 3.29 Gaussian probability density function.

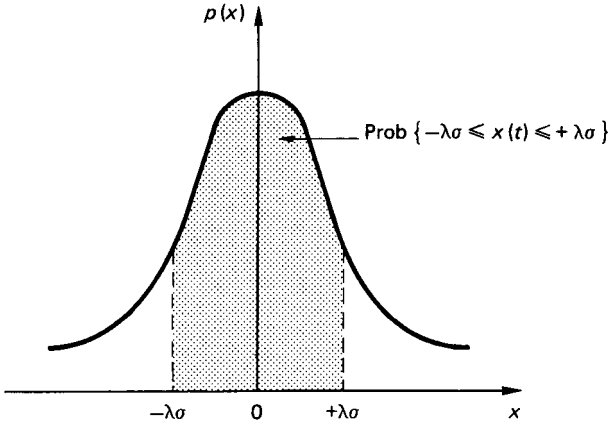


FIGURE 3.30 Gaussian probability density function with zero mean.

$$R(\tau) = \int_{-\infty}^{\infty} S(\omega) e^{-i\omega\tau} d\omega$$

If $\tau = 0$

$$R(0) = \int_{-\infty}^{\infty} S(\omega) d\omega$$

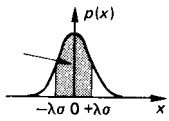
$$= E[x^2]$$

That is, the mean square value of a stationary random process x is the area under

TABLE 3.4

Value of λ	Prob$[-\lambda\sigma \leq x(t) \leq \lambda\sigma]$	Prob$[\lvert x(t)\rvert > \lambda\sigma]$
0	0	1.0000
0.2	0.1585	0.8415
0.4	0.3108	0.6892
0.6	0.4515	0.5485
0.8	0.5763	0.4237
1.0	0.6827	0.3173
1.2	0.7699	0.2301
1.4	0.8586	0.1414
1.6	0.8904	0.1096
1.8	0.9281	0.0719
2.0	0.9545	0.0455
2.2	0.9722	0.0278
2.4	0.9836	0.0164
2.6	0.9907	0.0093
2.8	0.9949	0.0051
3.0	0.9973	0.0027
3.2	0.9986	0.00137
3.4	0.9993	0.00067
3.6	0.9997	0.00032
3.8	0.9998	0.00014
4.0	0.9999	0.00006

Prob$[-\lambda\sigma \leq x(t) \leq +\lambda\sigma]$



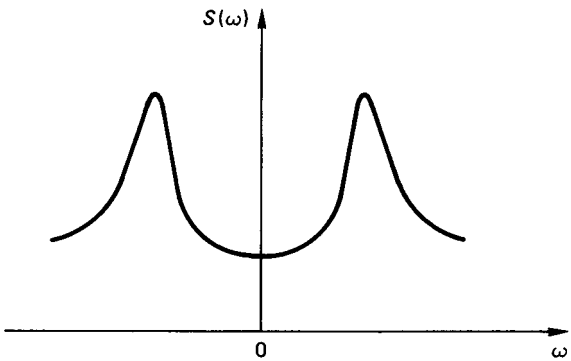


FIGURE 3.31 Typical spectral density function.

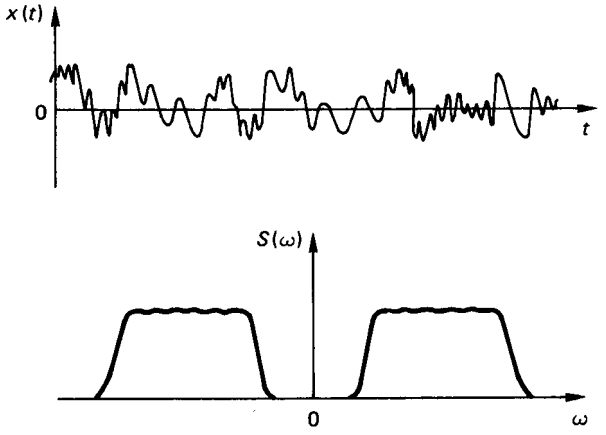


FIGURE 3.32 Wide-band process.

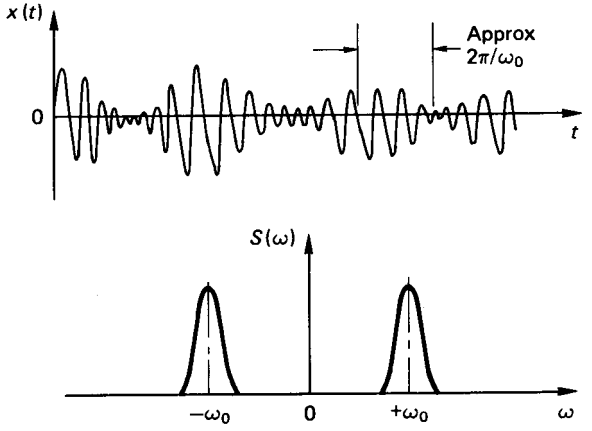


FIGURE 3.33 Narrow-band process.

the $S(\omega)$ against frequency curve. A typical spectral density function is shown in Figure 3.31.

A random process whose spectral density is constant over a very wide frequency range is called *white noise*. If the spectral density of a process has a significant value over a narrower range of frequencies, but one which is nevertheless still wide compared with the center frequency of the band, it is termed a *wide-band* process (Figure 3.32). If the frequency range is narrow compared with the centre frequency it is termed a *narrow-band process* (Figure 3.33). Narrow-band processes frequently occur in engineering practice because real systems often respond strongly to specific exciting frequencies and thereby effectively act as a filter.

PART 5
MECHANICS OF FLUIDS

Peter Tucker

3.10 INTRODUCTION

Fluid is one of the two states in which matter can exist, the other being solid. In the fluid state the matter can flow; it will, in general, take the shape of its container. At rest a fluid is not able to sustain shear forces.

Some ''solids'' may flow over a long period (glass window panes thicken at the base after a long time in a vertical position). The substances considered in fluid mechanics are those that are continuously fluid.

Fluid mechanics is a study of the relationships between the effects of forces, energy and momentum occurring in and around a fluid system. The important properties of a fluid in fluid mechanics terms are *density, pressure, viscosity, surface tension,* and, to some extent, *temperature,* all of which are intensive properties.

Density is the mass per unit volume of the substance. *Pressure* is the force per unit area exerted by the fluid on its boundaries. *Viscosity* is a measure of the fluid's resistance to flow and may be considered as internal friction. The higher the coefficient of viscosity, the greater the resistance. *Surface tension* is a property related to intermolecular attraction in the free surface of a liquid resulting in the apparent presence of a very thin film on the surface. The meniscus at the intersection of a liquid and its container wall and capillarity are further examples of intermolecular attraction.

Temperature is more relevant to thermodynamics than to fluid mechanics. It indicates the state of thermal equilibrium between two systems or, more loosely, the level of thermal energy in a system.

3.11 FLUID STATICS

Pressure at a Depth

The variation of pressure p and depth h in a fluid of density ρ is given by

$$\int_{p_1}^{p_2} dp = \int_{h_1}^{h_2} \rho g dh \tag{3.5}$$

Most liquids are assumed to be of constant density ρ. In such a liquid the pressure at a depth h below a free surface is given by

$$p = p_0 + \rho g h \tag{3.6}$$

where p_0 is the pressure above the free surface.

For gases equation (3.5) may be solved only if the relationship between ρ and h is known. A typical case is the atmosphere, where the relationship may be taken as polytropic or isothermal, depending on the altitude. Tables relating the properties of the atmosphere to altitude are readily available as the International Atmosphere.

Pressure Measurement

Pressure may be expressed as a pressure p in Pa, or as a pressure head h in m of the fluid concerned. For a fluid of density ρ, $p = \rho g h$. There are various instruments used to measure pressure.

Manometers. Manometers are differential pressure-measuring devices, based on pressure due to columns of fluid. A typical U-tube manometer is shown in Figure 3.34(a). The difference in pressure between vessel A containing a fluid of density ρ_A and vessel B containing fluid of density ρ_B is given by

$$p_A - p_B = \rho_B g Z_B + (\rho_m - \rho_B) g h - \rho_A g Z_A \tag{3.7}$$

where h is the difference in the levels of the manometer fluid of density ρ_m and $\rho_m > \rho_A$ and $\rho_m > \rho_B$. If $\rho_A = \rho_B = \rho$, then the difference in pressure head is

$$\frac{p_A - p_B}{\rho g} = Z_B - Z_A + \left(\frac{\rho_m}{\rho} - 1\right) h \tag{3.8}$$

If $\rho_m < \rho_A$ and $\rho_m < \rho_B$ then an inverted U-tube manometer is used as shown in Figure 3.34(b). In this case the pressure difference is

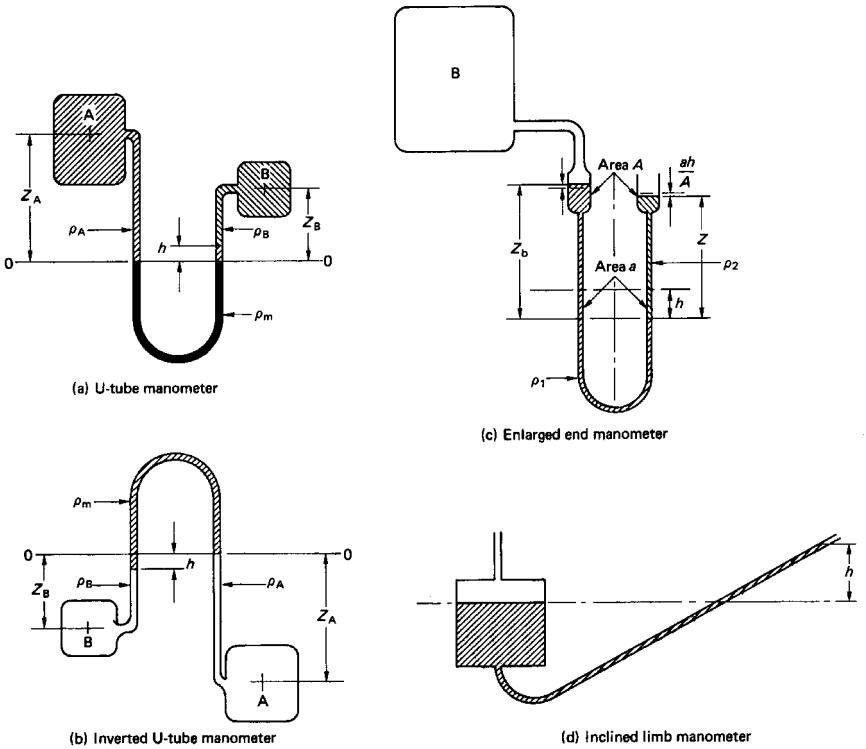


FIGURE 3.34

$$p_A - p_B = \rho_A g Z_A + (\rho_B - \rho_m) g h - \rho_B g Z_B \tag{3.9}$$

and if $\rho_A = \rho_B = \rho$ then the difference in pressure head is

$$\frac{p_A - p_B}{\rho g} = Z_A - Z_B + \left(1 - \frac{\rho_m}{\rho}\right) h \tag{3.10a}$$

of if $\rho_m << \rho$ (the manometer fluid a gas and A and B containing liquid),

$$\frac{p_A - p_B}{\rho g} = Z_A - Z_B + h \tag{3.10b}$$

The accuracy of a U-tube manometer may be increased by sloping one of the legs to increase the movement of the fluid interface along the leg for a given difference in vertical height. This may be further enhanced by replacing the vertical leg by a reservoir and the inclined leg by a small-bore tube (Figure 3.34(d)).

Another method is to increase the cross-sectional area of the ends of the legs (or one of the legs), as shown in Figure 3.34(c), so that a small movement of the free surfaces in the enlarged ends results in a large movement of the surface of separation.

Dial Gauges. Most pressure dial guages make use of a *Bourdon tube*. This is a curved tube with an oval cross section. Increase in pressure causes the tube to straighten, decrease makes it bend. The movement of the free end turns a pointer over a scale, usually via a rack-and-pinion mechanism. The scale may be calibrated in the required pressure units.

Diaphragm Gauges. In these gauges the pressure changes produce a movement in a diaphragm which may be detected by a displacement transducer, or by the output from strain gauges attached to the diaphragm surface.

Piezoelectric Transducers. A piezoelectric crystal produces a voltage when deformed by an external force. This induced charge is proportional to the impressed force and so the output can be used to supply a signal to a measuring device which may be calibrated in pressure units.

Fortin Barometer. Barometers are used to measure the ambient or atmospheric pressure. In the Fortin barometer a column of mercury is supported by the atmospheric pressure acting on the surface of the mercury reservoir. The height h of the column above the reservoir surface, usually quoted as millimeters of mercury (mm Hg), may be converted to pressure units p_0 by

$$\begin{aligned} p_0 &= \rho g h = 13.6 \times 9.81 h \\ &= 133.42 h \text{ Pa} \end{aligned} \tag{3.11}$$

Aneroid Barometer. In this device the atmospheric pressure tends to compress an evacuated bellows against the elasticity of the bellows. The movement of the free end of the bellows drives a pointer over a dial (or a pen over a drum graph) to indicate (or record) atmospheric pressure variations.

Buoyancy

When a body is immersed in a fluid the difference in pressure over the depth of the body produces a displacement force on the body. The first recognition of this is attributed to Archimedes.

Displacement Force. The buoyancy or displacement force F_B on a body fully or partially immersed in a fluid is equal to the weight of the volume of the fluid equivalent to the immersed volume of the body (the weight of the displaced volume V_D of the fluid):

$$F_B = \rho g V_D \tag{3.12}$$

This buoyancy force acts vertically upwards through the centroid of the displaced volume, which is known as the *center of buoyancy* (B). If the buoyancy force is equal to the weight of the body then the body will float in the fluid. If the weight of the body is greater than the buoyancy force then the body will sink. If the buoyancy force is greater than the weight of the body then the body will rise.

In a liquid, for example, a body will sink until the volume of liquid displaced has a weight which is equal to that of the body. If the body is more dense than the liquid then the body will not float at any depth in the liquid. A balloon will rise in air until the density of the air is such that the weight of the displaced volume of air is equal to the weight of the balloon.

Stability of a Floating Body. Figure 3.35 shows bodies in various stages of equilibrium. A body is in *stable* equilibrium if a small displacement produces a restoring force or moment as for the ball in the saucer in Figure 3.35(a) or the floating bodies in (d) and (g). A body is in *unstable* equilibrium if a small displacement produces a disturbing force or moment as for the ball in Figure 3.35(b) or the floating bodies (e) and (h). A body is in *neutral* equilibrium if a small displacement produces no force or moment as for the ball in Figure 3.35(c) or the floating bodies in (f) and (i).

For a partially immersed body, the point at which the line of action of the buoyancy force F_B cuts the vertical center line of the floating body in the displaced position is known as the *metacenter* (M). For a floating body to be stable M must lie above the body's center of gravity, G. If M lies below G the body is unstable; if M lies on G the body is in neutral equilibrium. The distance GM is known as the *metacentric height*. The distance of the metacenter above the center of buoyancy

$$BM = \frac{I}{V_D} \tag{3.13}$$

where I second moment of area of the body at the water line (liquid surface) about its central axis normal to the direction of displacement. (See Table 3.5.)

Period of Oscillation of a Stable Floating Body. A floating body oscillates with the periodic time T of a simple pendulum of length k^2/GM, where k is the radius of gyration of the body about its axis of rotation. The periodic time is given by

$$T = 2\pi \left(\frac{k^2}{GMg} \right)^{0.5} \tag{3.14}$$

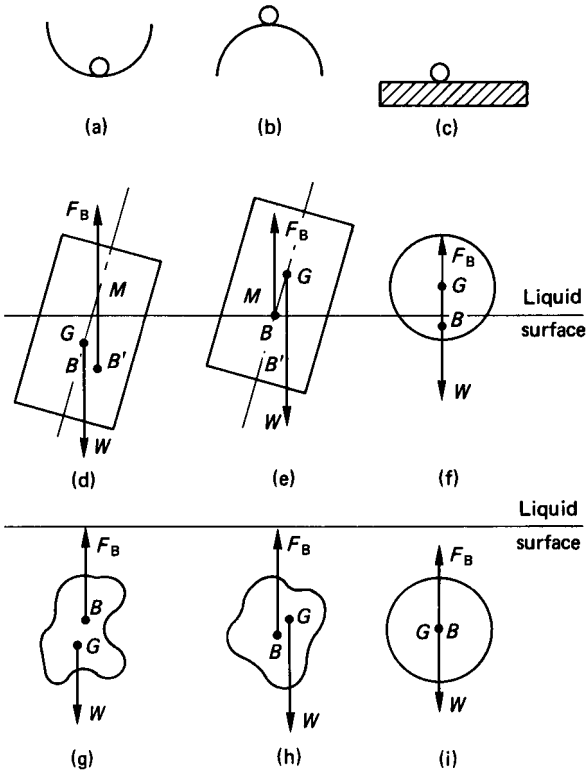


FIGURE 3.35 Stability.

3.12 FLUID FLOW

Definitions

Continuity. For almost all analysis, a fluid is considered to be a *continuum*, that is, with nondiscontinuities or cavities in the flow stream. Cavitation, two-phase flow, "bubbly" flow, etc. are special cases with nonstandard relationships.

Therefore for a continuum, by considering the flow through an elemental cuboid the *continuity equation* in three dimensions may be shown to be

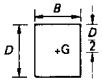
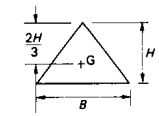
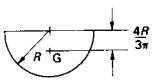
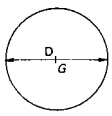
$$\frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = 0 \tag{3.15}$$

where v_x is the fluid velocity in the x direction, etc. For a fluid of constant density

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 \tag{3.16}$$

That is, the velocity of an incompressible fluid flow cannot increase in all three directions at the same time without producing discontinuity or cavitation.

TABLE 3.5 Second Moments of Area

	Parallel axis theorem $I_X = I_G + A\bar{x}^2$	
	Area A	I_G
	BD	$\frac{BD^3}{12}$
	$\frac{BH}{2}$	$\frac{BD^3}{36}$
	$\frac{\pi R^2}{2}$	$0.1102R^4$
	$\frac{\pi D^2}{4}$	$\frac{\pi D^4}{64}$

For two-dimensional flow:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{3.17}$$

For one-dimensional flow the continuity equation may be linked with the *conservation of mass,* which states that for steady flow conditions mass flow rate, $\dot{m}$, is constant throughout a flow system:

$$\dot{m} = \rho A v \tag{3.18}$$

where A is the cross-sectional area normal to the direction of flow.

***Circulation* Γ.** Circulation is defined as the line integral of the tangential velocity around a closed contour:

$$\Gamma = \oint v_s ds \tag{3.19}$$

Γ is positive if the closed contour is on the left.

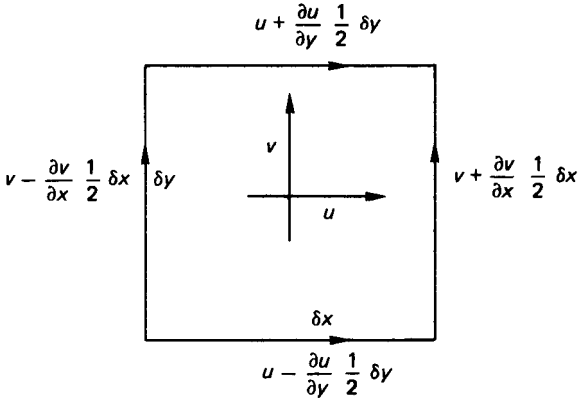
Vorticity ζ. Vorticity is defined as the circulation per unit area, and by considering the circulation around the element in Figure 3.36(a) it can be shown that

$$\zeta = \frac{\Gamma}{\partial x \partial y} = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \tag{3.20}$$

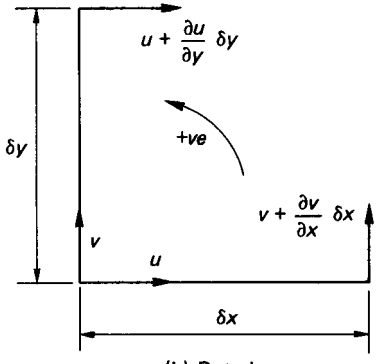
Rotation ω. Rotation is defined as the instantaneous mean angular velocity of two mutually perpendicular lines in a plane of the flow field. By considering the angular velocities of the two lines *OA* and *OB* in Figure 3.36(b) it can be shown that

$$\omega = \frac{1}{2}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) \tag{3.21}$$

or the rotation is equal to half the vorticity.



(a) Vorticity



(b) Rotation

FIGURE 3.36

Stream Lines. The stream line is a line drawn in a flow stream which is everywhere tangential to the direction of flow. A family of stream lines may be described mathematically by a stream function ψ, where ψ fn(x,y). Each stream line has the same function with a value of ψ peculiar to that line.

Stream Tubes. Since a line has no thickness, there can be no flow along a stream line. The stream tube is a concept introduced to enable flow along a stream line to be studied. It is a tube of infinitely small cross-section with a stream line as its axis.

Energy. Energy is the stored form of *heat* and *work*. The basic concepts applied in fluid mechanics are:

- The conservation of energy
- That energy is transferred only as heat or work
- That energy in a fluid flow system is stored only as *internal energy, kinetic energy,* or *potential energy*

Other forms of energy (electrical, magnetic, chemical, etc.) may have to be taken into account in some circumstances, but are not usually included in general fluid mechanics relationships.

Enthalpy and *entropy* need to be considered for gas flow analysis. The basic energy-flow equation is the *steady-flow energy equation:*

$$\dot{Q} + \dot{W} = \dot{m}\Delta\left(h + \frac{v^2}{2} + gZ\right) \tag{3.22}$$

where $\dot{Q}$ = the rate of heat transfer
$\dot{W}$ = the rate of work transfer (power)
h = the specific enthalpy (if e is the specific internal energy, p the pressure and ρ the fluid density, then $h = e + (p/\rho)$)
Z = the height above some datum
v = the mean velocity of flow

Specific means "per unit mass." For non-steady flow conditions, either *quasi-steady* techniques or the integration of infinitely small changes may be employed.

Momentum. Momentum is the product of mass and velocity (mv). Newton's laws of motion state that the force applied to a system may be equated to the rate of change of momentum of the system, in the direction of the force. The change in momentum may be related to time and/or displacement. In a steady flow situation the change related to time is zero, so the change of momentum is usually taken to be the product of the mass flow rate and the change in velocity with displacement. Hence, the force applied across a system is

$$F = \dot{m}\Delta v \tag{3.23}$$

where Δv is the change in velocity in the direction of the force F.

For flow in two or three dimensions the resultant force may be obtained by resolving the forces in the usual way. The flow round an expanding bend shown in Figure 3.37 is a typical example. The force in the x direction, F_x, and the force in the y direction, F_y, are given by

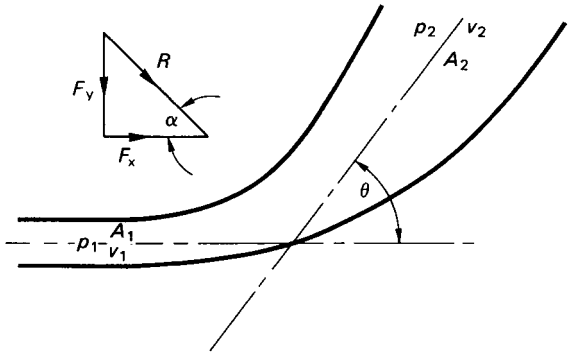


FIGURE 3.37 Expanding bend.

$$F_x = p_1A_1 + \dot{m}v_1 - (p_2A_2 + \dot{m}v_2)\cos\theta \tag{3.23a}$$

$$F_y = -(p_2A_2 + \dot{m}v_2)\sin\theta \tag{3.23b}$$

from which the resultant force F_R and its angle of inclination α can be found:

$$F_R = \{F_x^2 + F_y^2\}^{0.5} \tag{3.24}$$

and

$$\alpha = \tan^{-1}\left(\frac{F_y}{F_x}\right) \tag{3.25}$$

In pipe flow the pressure forces pA must not be forgotten.

Equations of Motion

Application of the momentum equation in three dimensions to an irrotational, inviscid fluid flow leads to the *Euler equation:*

$$-\frac{Dv}{Dt} = \frac{1}{\rho}\nabla p + g\nabla h \tag{3.26}$$

which for steady flow along a stream tube becomes:

$$\frac{1}{\rho}dp + vdv + gdZ = 0 \tag{3.27}$$

Integration for a constant-density fluid gives:

$$\frac{p}{\rho} + \frac{v^2}{2} + gZ = \text{constant} \tag{3.28}$$

These energy per unit mass terms may be converted to energy per unit weight terms, or heads, by dividing by g to give:

$$\frac{p}{\rho g} + \frac{v^2}{2g} + Z = \text{constant} \tag{3.29}$$

which is the *Bernoulli* (or constant head) equation.

These equations are the generally more useful simplifications of the *Navier–Stokes* equation:

$$\frac{Dv}{Dt} = \rho B - \nabla p + \nabla\{u(\nabla v + \nabla \epsilon)\} \tag{3.30}$$

where B is the body force and ϵ the rate of expansion.

Incompressible Pipe Flow

Flow Regimes. The two major flow regimes are *laminar* and *turbulent*. Laminar flow may be fairly accurately modelled mathematically. The fluid moves in smooth layers and the velocity is everywhere tangential to the direction of motion. Any perturbations are quickly dampened out by the fluid viscosity.

In turbulent flow the mathematical models usually need to be empirically modified. Viscous damping may not be sufficient to control the perturbations, so that the fluid does not move in smooth layers and the instantaneous velocity may have components at an angle to the direction of motion.

The ratio of inertia forces to viscous forces in a fluid flow is known as *Reynolds' number* (*Re*). In a pipe diameter D, with a fluid of density ρ and dynamic viscosity η flowing with velocity v, Reynolds' number $Re = \rho Dv/\eta$.

A high value of $Re > 2300$ indicates relatively low damping, predicting turbulent flow. A low value of $Re < 2000$ indicates relatively high damping, predicting laminar flow. These values were suggested in an historical experiment by Osborne Reynolds.

Pipe Losses (friction). Liquids (and gases under small pressure changes) flowing through pipes usually behave as incompressible fluids. Within the flow there is a relationship between the shear stress in the fluid and the gradient of the change of velocity across the flow. In most light liquids and gases, the relationship approximates to the *Newtonian* one:

$$\tau = \eta \frac{dv}{dy} \tag{3.31}$$

where τ is the shear stress in the fluid, dv/dy the gradient of the velocity distribution across the pipe and η the dynamic viscosity.

The viscosity of the fluid produces not only the velocity variation across the flow but also a loss of useful energy along the pipe usually regarded as a friction loss. The force associated with this loss of energy appears as a shear force in the fluid at the pipe wall. A relationship between the shear stress at the pipe wall τ_0 and the *friction coefficient, f* is:

$$\tau_0 = \frac{1}{2} \rho v^2 f \tag{3.32}$$

where v is the average flow velocity.

For use in pipe flow problems with viscous fluids the Bernoulli equation (3.29) may be adapted to include a head loss term, h_L. Applied between two positions (1) and (2) in a pipe in a gravitational field of acceleration g, the head equation gives:

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + Z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + Z_2 + h_L \tag{3.33}$$

where the head loss term h_L is the loss of energy per unit *weight* of fluid flowing.

Note that if a pump, say, is introduced between (1) and (2) an energy gain per unit weight term h_w, equivalent to the output of the pump written as a head, should be added to the left-hand side of the equation to give

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + Z_1 + h_w = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + Z_2 + h_L \tag{3.34}$$

The relationship used to determine the head loss in a pipe depends on the flow regime in operation as well as the type and surface finish of the pipe wall.

A mathematical analysis of laminar flow may be used to obtain an expression for the head loss along a pipe in terms of the fluid properties, pipe dimensions and flow velocity. Relating the pressure change along a length, L, of pipe of diameter, D, to the change in shear force across the flow produces *Poiseuille's equation:*

$$h_L = 32\,\frac{\eta v L}{\rho g D^2} \tag{3.35}$$

If the flow regime is turbulent, then the relationships in the flow cannot be easily described mathematically, but the head loss may be derived by equating the shear force at the pipe wall to the change in pressure force along the pipe. This gives the *D'Arcy equation:*

$$h_L = \frac{4fL}{D}\,\frac{v^2}{2g} \tag{3.36}$$

This relationship may also be established using dimensional analysis.

Unfortunately, the friction coefficient, f is *not* a constant but depends on the type of flow and the roughness of the pipe walls. There are general relationships between f and Re which may be expressed as equations of varying complexity or as charts. For smooth pipes:

$$\frac{1}{\sqrt{f}} = 4\,\log_{10}\,(2R_e\,\sqrt{f}) - 1.6 \tag{3.37}$$

For rough pipes with a roughness size k this becomes:

$$\frac{1}{\sqrt{f}} = 4\,\log_{10}\left(\frac{D}{2k}\right) + 3.48 \tag{3.38}$$

The *Colebrook* and *White equation* is a general or universal friction equation:

$$\frac{1}{\sqrt{f}} = 3.47 - 4\,\log_{10}\left(\frac{2r}{D} + \frac{9.35}{Re\sqrt{f}}\right) \tag{3.39}$$

It is, however, usually more useful to obtain values of f from a chart such as Figure

3.38. (*Note:* the value of f used in American equations for head losses is *four times* that used in the United Kingdom, so if values of f are obtained from American texts they should be moderated accordingly or the corresponding American equation used.)

An empirical relationship widely used in water pipe work is the *Hazen–Williams equation,* usually written as:

$$v = 1.38\ C\ m^{0.63} \left(\frac{h_L}{L}\right)^{0.54} \tag{3.40}$$

where m is the ratio of the cross-sectional area of flow to the wetted perimeter known as the *hydraulic mean diameter* and C is a coefficient which depends on the condition of the pipe wall.

Pipe Losses (Changes in Section). When a fluid flows through a sharp (sudden) change in the cross-section of a pipe, energy is dissipated in the resulting turbulent eddies at the edge of the flow stream, producing a loss of head (or energy per unit weight). If the flow is from a smaller area to a larger one (sudden enlargement), the head loss is

$$h_L = \frac{(v_1 - v_2)^2}{2g} \tag{3.41}$$

When the flow is from a larger area to a smaller area (sudden contraction), the narrowed flow stream entering the smaller pipe is known as a *vena contracta*. The loss of head is assumed to be that due to a sudden enlargement from the vena contracta to the full area of the smaller pipe:

$$h_L = \frac{v_2^2}{2g}\left(\frac{1}{C_c} - 1\right)^2 \tag{3.42}$$

The *contraction coefficient*, C_c, is the ratio of the vena contracta area to that of the smaller pipe area. A typical value of C_c is 0.6, which gives

$$h_L \simeq = 0.5\ \frac{v_2^2}{2g} \tag{3.43}$$

which is also the head loss at the sharp entry to a pipe from a reservoir. Energy dissipation at changes in section, and pipe entry and exit, may be reduced by making the changes smooth and gradual, though this may be relatively costly.

Other pipe fittings, such as valves, orifice plates and bends, produce varying values of head loss, usually quoted as a fraction of the velocity head ($v^2/2g$).

Pipe Networks. A system of pipes may be joined together either in series (one after the other) or parallel (all between the same point). The friction head loss across a system of pipes in series is the sum of the losses along each pipe individually. The flow rate through each pipe will be the same. Using D'Arcy's head loss equation:

$$h_L = 4f_1 \frac{L_1 v_1^2}{D_1 2g} + 4f_2 \frac{L_2 v_2^2}{D_2 2g} + \ldots 4f_n \frac{L_n v_n^2}{D_n 2g} \tag{3.44}$$

and

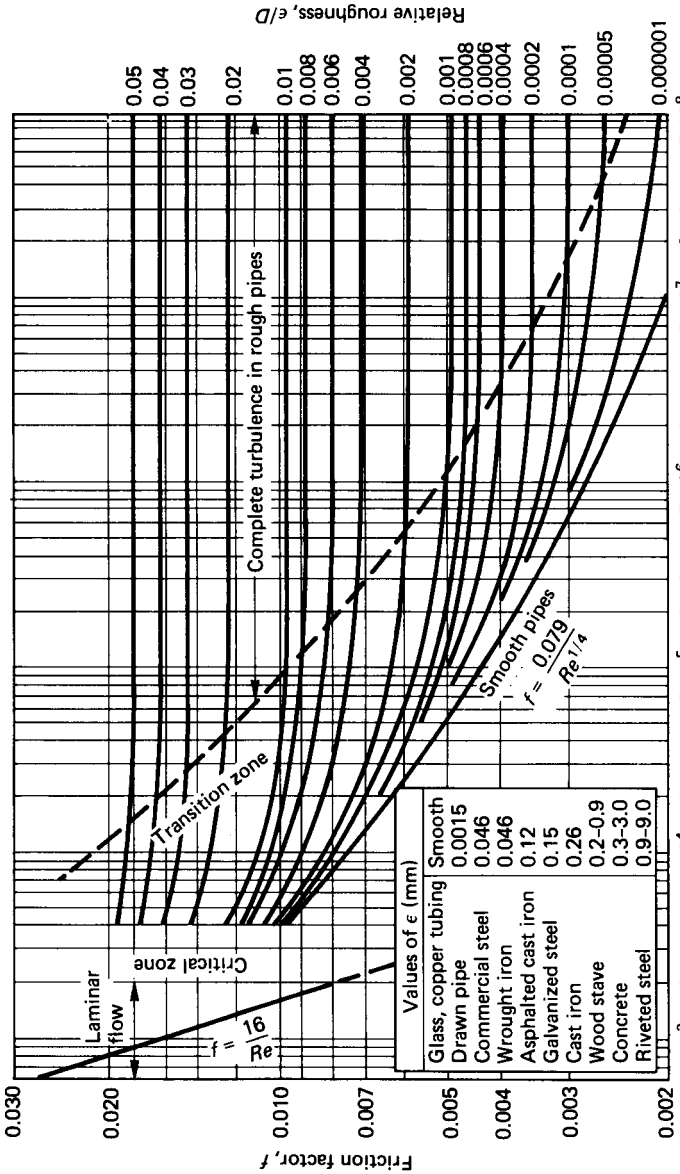


FIGURE 3.38

$$\dot{V} = v_1 A_1 = v_2 A_2 = \cdots = v_n A_n \tag{3.45}$$

If the system of pipes is connected in parallel the head loss across the system is equal to the head loss along any one of the pipes, when the flow has settled down to steady. The flow rate through the system is the sum of the flow rates along each pipe. Again using the D'Arcy equation:

$$h_L = 4f_1 \frac{L_1 v_1^2}{D_1 2g} = 4f_2 \frac{L_2 v_2^2}{D_2 2g} = \cdots = 4f_n \frac{L_n v_n^2}{D_n 2g} \tag{3.46}$$

$$\dot{V} = v_1 A_1 + v_2 A_2 + \ldots v_n A_n \tag{3.47}$$

In addition, the rate of flow into each junction of a network, either in series or parallel, is equal to the rate of flow out of it.

Pipe network problems are thus solved by setting up a number of such equations and solving them simultaneously. For a large number of pipes a computer program may be needed to handle the number of variables and equations. An example of a pipe network computer solution is given in Douglas et al. (1986).

3.13 FLOW MEASUREMENT

Pipe Flow

Orifices and Nozzles (see Figure 3.39(a)). Another basic flow measurement technique is to introduce some restriction into the flow passage and calibrate the resulting pressure changes against known flow rates.

Often the restriction in a pipe is in the form of an *orifice plate* (a plate with a hole) or a *nozzle*. A simple application of the Bernoulli equation may be used for the design calculations, but it is always advisable to calibrate any measurement device in conditions as close to the required operating conditions as possible.

Bernoulli and the continuity equations give the flow rate:

$$\dot{V} = C_d A_o \left\{ \frac{2(p_p - p_o)}{\rho[1 - (A_o/A_p)^2]} \right\}^{0.5} \tag{3.48}$$

where A_o = the orifice (or nozzle throat) area
A_p = the upstream pipe area
p_p = the upstream pressure
p_o = the pressure at the orifice or the nozzle throat
C_d = a discharge coefficient which takes account of losses and contraction of the flow stream through the device

Recommended orifice and nozzle dimensions, values of C_d and methods of operation are contained in BS 1042. It is most important to place the orifice or nozzle so that its operation is not affected by perturbations in the upstream flow caused by valves, bends or other pipe fittings.

Venturi Meters (see Figure 3.39(b)). The introduction of any restriction, particularly a sharp-edged orifice or nozzle, in a pipe will result in a loss of head. If it is required to keep this loss to a minimum, a *Venturi meter* may be used. The flow

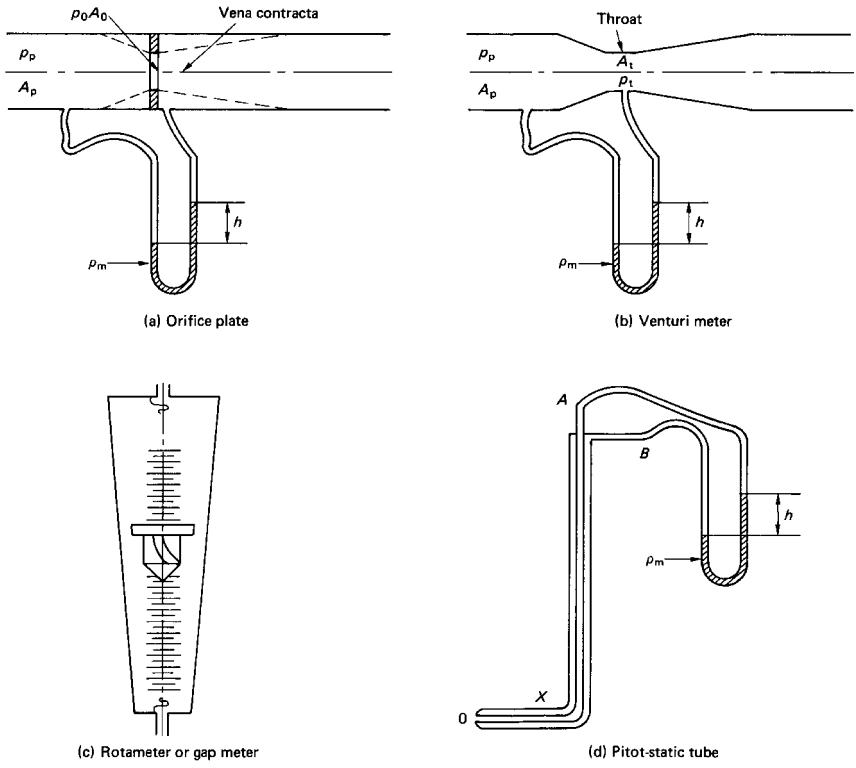


FIGURE 3.39 Flow meters.

passage in a Venturi is gradually and smoothly reduced to a throat followed by a controlled expansion to full pipe section. In this way the head loss across the meter is greatly reduced, but the cost of producing a Venturi meter is much higher than that of an orifice. Equation (3.48) may be used to calculate the flow rate $\dot{V}$, but the value of C_d will now be approximately 0.98 for a well-designed Venturi meter. Again, BS 1042 should be consulted for recommended dimensions, values of C_d and methods of operation.

Rotameter or Gap Meter (see Figure 3.39(c)). If, somewhere within the system, it is acceptable to tolerate flow up a vertical section of piping, then a *rotameter or gap meter* may be used. This instrument depends on the balancing of the weight of a rotating float in a tapered glass tube with the drag forces in the annular passage surrounding the float. The drag forces depend on the flow rate and the corresponding area of the annulus. As the flow rate increases, the annulus area which will produce a drag force equal to the weight of the float also increases. Therefore the float moves up the tapered tube until the annulus area is such that the forces again balance. As the flow rate decreases the float descends to a reduced annulus area to again achieve a balance of forces.

Velocity Meter. These are devices that measure velocity and not flow rate directly. *Pitot* and *Pitot-static* tubes are examples of such velocity-measuring instruments, making use of the pressure difference between the undisturbed flow stream and a point where the flow velocity is zero. They consist of two concentric tubes bent into an L shape as in Figure 3.39, with the outer tube joined to the inner at the toe of the *L*, at *O*. This end is usually spherical with a hole through to the inner tube. The undisturbed flow is assumed to be in the region of the holes round the periphery of the outer tube at *X*. The velocity is assumed zero at the spherical end presented to the flow, at *O*.

The flow velocity, v may be calculated by applying Bernoulli's equation between the two points *O* and *X* to give

$$v = C_v \left[2 \frac{(p_o - p_x)}{\rho} \right]^{0.5} \tag{3.49}$$

where p_o is connected to *O* via the inner tube to the tapping at *A*, p_x is connected to *X* via the outer tube to the tapping at *B* and C_v is a coefficient to cater for losses and disturbances not
accounted for in Bernoulli's equation. C_v is often taken to be unity. The pressure difference may be measured using a manometer and then written into equation (3.49) as a head, h, to give

$$v = \left[2gh \left(\frac{\rho_m}{\rho} - 1 \right) \right]^{0.5} \tag{3.50}$$

As usual, it is advisable to calibrate the tube and obtain a calibration curve or an accurate value for C_v.

Care should be taken when a pitot-static tube is used to measure pipe flow, since the velocity will vary across the pipe. As a rough guide to the flow rate, the maximum velocity, which is at the center of the pipe, may be taken to be twice the average velocity. Alternatively, the velocity at half the radius may be taken to be equal to the average velocity in the pipe. For an accurate evaluation the velocity distribution curve may be plotted and the flow rate through the pipe found by integration. This may be approximated to by dividing the cross-section into a series of concentric annuli of equal thickness, measuring the velocity at the middle of each annulus, multiplying by the corresponding annulus area, and adding to give the total flow rate.

Current meters, torpedo-shaped devices with a propeller at the rear, may be inserted into pipes. The number of rotations of the propeller are counted electrically. This number together with coefficients peculiar to the propeller are used in empirical equations to determine the velocity.

Velometers, vaned anemometers, and *hot wire anemometers* are not usually used to measure the velocities of incompressible fluids in pipes, and will be discussed below under Gas Flow.

3.14 BOUNDARY LAYER FLOW

When a fluid flows over a solid boundary there is a region close to the boundary in which the fluid viscosity may be assumed to have an effect. Outside this region the fluid may be assumed inviscid. The viscous effect within the region is evidenced

by a reduction in velocity as the boundary is approached. Outside the region the velocity is constant. The region is known as a *boundary layer*.

It is usual to assume that at the solid surface the fluid velocity is zero and at the boundary layer outer edge it is equal to the undisturbed flow velocity v_s. This defines the boundary layer thickness δ. (In practice, δ may be taken to be the distance from the boundary surface at which the velocity is 99% of the undisturbed velocity, or 0.99 v_s.)

When a flow stream at a velocity v_s passes over a flat plate, the boundary layer thickness δ is found to increase with the distance x along the plate from the leading edge. Near the leading edge the flow inside the boundary layer may be assumed to be laminar, but as x increases the flow becomes turbulent and the rate of increase of δ with x also increases, as shown in Figure 3.40.

Within even a turbulent boundary layer there is a narrow region close to the plate surface where the flow is laminar. This is known as the *viscous sublayer* and has thickness δ_b. The reduction in velocity across the boundary layer is associated with a shear force at the plate surface, usually known as the *drag force*.

Application of the momentum equation produces *Von Karman's momentum integral*, in which the drag force per unit width, F_D, becomes

$$F_D = \rho v_s^2 \int_0^\delta \frac{v}{v_s}\left(1 - \frac{v}{v_s}\right) dy \tag{3.51}$$

where v is the velocity within the boundary layer at a distance y above the plate surface. (The integral

$$\int_0^\delta \frac{v}{v_s}\left(1 - \frac{v}{v_s}\right) dy$$

may be defined as the *momentum thickness* (θ) and the integral

$$\int_0^\delta \left(1 - \frac{v}{v_s}\right) dy$$

as the *displacement thickness* (δ^*) so that

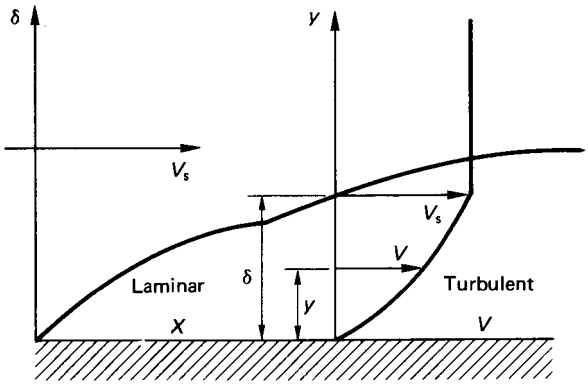


FIGURE 3.40 Boundary layer.

$$F_D = \rho v_s^2 \theta \tag{3.52}$$

In order to solve the Von Karman integral equation (3.51) or equation (3.52) it is necessary to know the value of δ and the relationship between v and y, the velocity distribution. Both of these are dependent on each other and the flow regime, laminar or turbulent, within the boundary layer.

Laminar Boundary Layers. A flat plate laminar boundary is normally assumed if $Re_x < 500\ 000$. (Re_x is Reynolds' number based on x or $\rho v_s x/\eta$.) For laminar boundary layers various simplified velocity distribution relationships may be used, such as linear, sinusoidal, or cosinsusoidal. The generally accepted most accurate relationship is, however, that obtained by the reduction of a four-term polynomial, which gives

$$\frac{v}{v_s} = \frac{3y}{2\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3 \tag{3.53}$$

From this the shear stress at the plate surface, τ_o, may be found for Newtonian fluids:

$$\tau_o = \eta \left(\frac{dv}{dy}\right)_{y=0} \tag{3.54}$$

The shear force

$$F_D = \int_0^x \tau_0 dx \tag{3.55}$$

Substitution from equation (3.53) in equations (3.51) and (3.54) and equating F_D from equations (3.51) and (3.55) leads via a separation of variables technique to

$$\frac{\delta}{x} = 4.64\ Re_x^{-0.5} \tag{3.56}$$

The drag force is usually quoted in terms of a *drag coefficient,* C_D:

$$C_D = \frac{2F_D}{\rho v_s^2 x} \tag{3.57}$$

or

$$F_D = \frac{1}{2} C_D \rho v_s^2 x \tag{3.58}$$

By manipulation of the above equations an equation for C_D for a laminar boundary over the whole length L of the plate:

$$C_D = 1.29 Re_L^{-0.5} \tag{3.59}$$

is obtained. The drag force on the whole plate surface of area A is found from:

$$F = \frac{1}{2} C_D v_s^2 A \tag{3.60}$$

Turbulent Boundary Layers

For Re_x 500 000, a flat plate boundary layer is assumed to be turbulent. In a turbulent boundary layer the velocity distribution is often written in a power form:

$$\frac{v}{v_s} = \left(\frac{y}{\delta}\right)^{1/n} \tag{3.61}$$

The index n varies between 6 and 9, depending on Re_x.

Because of the presence of the laminar sublayer, the turbulent regime is not continuous down to the plate surface, and $(dv/dy)_{y=0}$ does not give a useful result.

The equation used for τ_0 is

$$\tau_o = 0.0225\rho v_s^2 R_\delta^{-0.25} \tag{3.62}$$

based on work on smooth pipes by Blasius.

Taking $n = 7$ and using the same techniques as for laminar boundary layers gives:

$$\frac{\delta}{x} = 0.37\ Re_x^{-0.2} \tag{3.63}$$

and

$$C_D = 0.072\ Re_L^{-0.2} \tag{3.64}$$

This result assumes that the turbulent boundary layer obtains over the whole length of the plate to L.

Prandtl suggested a more realistic expression which takes into account the presence of a laminar boundary layer near the leading edge:

$$C_D = 0.074\ Re_L^{-0.2} - 1700\ Re_L^{-1} \tag{3.65}$$

This may be used for $5 \times 10^5 < Re_L < 10^7$. For $10^7 < Re_L < 10^9$ Schlichting (1960) suggests a logarithmic velocity distribution and

$$C_D = 0.44(\log_{10} Re_L)^{-2.58} = 3.91(\ln Re_L)^{-2.58} \tag{3.66}$$

Again, equation (3.60) may be applied to find the drag force on the whole plate.

Viscous Sublayers

The analyses above assume that the plate surface is smooth or at least *hydraulically smooth*. A surface is regarded as hydraulically smooth if the average roughness height k is less than the laminar sublayer thickness δ_b. For a turbulent layer with a velocity distribution power index of $n = 1/7$, the laminar sublayer thickness at a point at a distance x along the plate from the leading edge is given by

$$\frac{\delta_b}{\delta} = 199\ Re_x^{-0.7} \tag{3.67}$$

Thus δ_b may be compared with the roughness height, k, if the boundary layer thickness, δ, is known.

3.15 PRESSURE TRANSIENTS

Pressure transients may cause damage to piping systems such as aircraft fuel supply systems. It can be much more serious on a larger scale, where high-pressure rises over short periods may cause severe damage. Similar effects due to valve closure can be analysed on different levels of sophistication.

The simplest is the *rigid column* theory, which assumes that the fluid is incompressible, and that the valve is closed relatively slowly.

Slow Valve Closure

When a fluid flowing through a pipe with a velocity v_0 undergoes a change in velocity there is an associated change in pressure. Equating the force due to the pressure change to the rate of change of momentum during closure gives the resulting pressure rise Δp over a length of pipe L:

$$\Delta p = -\rho L \frac{dv}{dt} \tag{3.68}$$

The solution to this equation depends on a knowledge of the relationship between v and t (the valve closure rate in terms of the flow velocity).

Equation (3.68) is only applicable to relatively slow valve closure rates in which the closure time should not be less than $2L/C$ (where C is the speed of sound in the fluid).

Time to Establish Flow

The rigid-column theory is also often used to calculate the time required to establish flow in a pipe on opening a valve. The theory implies that the time required to fully establish the flow is infinite and so the time t to achieve 99% of the final velocity v_0 is usually accepted:

$$t = 2.646 \frac{Lv_0}{gH} \tag{3.69}$$

where H is the supply head to the pipe entrance. The time t_x required to reach $x\%$ of the final velocity is given by

$$t_x = \frac{Lv_0}{2gH} \ln\left(\frac{1 + 0.01x}{1 - 0.01x}\right) \tag{3.70}$$

Rapid Valve Closure

When a fluid is brought to rest *instantaneously* from a velocity of v_0 by the closure of a valve at the exit of a pipe of diameter D, there will be a relatively high pressure rise at the valve. If the valve closure time is *less than* $2(L/c)$ then the resultant pressure rise is as if it were instantaneous; c is the speed at which the pressure wave travels through the fluid, which is the sonic velocity.

On such a rapid valve closure the kinetic energy of the flow is converted into strain energy in both the pipe material and the fluid (even liquids are acknowledged as compressible in this context). The resulting pressure wave is transmitted through the fluid away from the valve as shown in Figure 3.41. The pressure rise produced is

$$\Delta p = \rho c v_0 \tag{3.71}$$

For a fluid of bulk modulus G, in a pipe of wall thickness x, of a material with a Young's modulus E and Poisson's ratio σ, the velocity of the pressure wave is

$$c = \left[\rho \left\{ \frac{1}{G} + \frac{D}{Ex} (1.25 - \sigma) \right\} \right]^{-0.5} \tag{3.72}$$

or

$$c = \left[\rho \left(\frac{1}{G} + \frac{D}{Ex} \right) \right]^{-0.5} \tag{3.73}$$

if longitudinal stress is small compared to hoop stress.

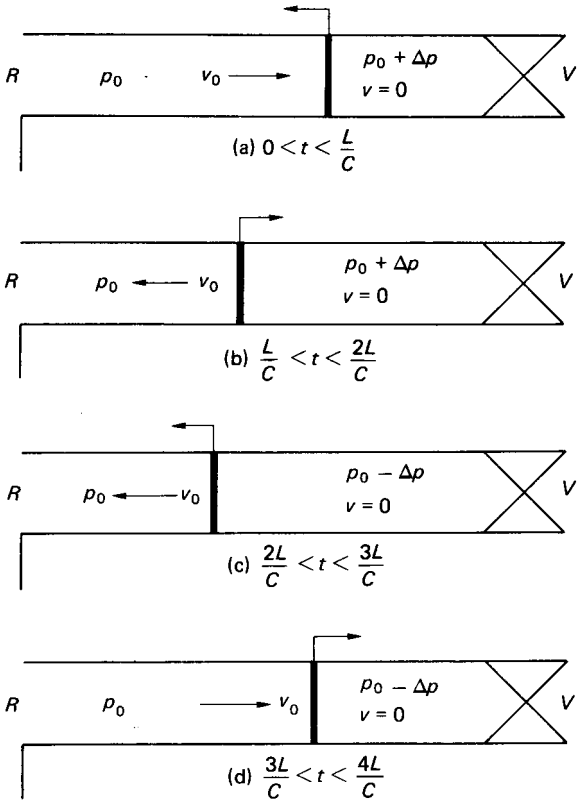


FIGURE 3.41 Progress of a pressure wave.

Equation (3.72) is often written as $c = [G_e/\rho]^{0.5}$, where G_e is the effective bulk modulus of the fluid and pipe combinations:

$$\frac{1}{G_e} = \frac{1}{G} + \frac{D}{Ex}(1.25 - \sigma) \tag{3.74}$$

If the valve is at the entrance to the pipe, then rapid valve closure results in a rarefaction (pressure drop) at the valve. In other words, the pressure change is

$$\Delta p = -\rho c v_0 \tag{3.75}$$

The Progress of a Pressure Wave

Assuming no friction and no cavitation in the fluid, the progress of a pressure wave along a pipe between a valve and a reservoir following valve closure is as shown in Figure 3.41. The fluid in the pipe is successively brought to rest by the passage of the pressure wave.

At a time $t = L/c$ after valve closure, the pressure wave reaches the reservoir. The whole of the fluid in the pipe is at rest at a pressure $p = p_0 + \Delta p$, which at the reservoir end instantaneously drops to reservoir pressure p_0. The resulting pressure wave travels along the pipe towards the valve and the fluid at the higher pressure in the pipe flows towards the reservoir at its initial velocity v_0.

At $t = 2L/c$, the situation is the same as for a rapid closure of a valve downstream of the flow, producing an instantaneous pressure drop to $p_0 - \Delta p$ and a rarefaction which travels towards the reservoir. The passage of the rarefaction successively brings the fluid to rest along the pipe.

At $t = 3L/c$, the rarefaction reaches the reservoir and the pressure instantaneously rises to reservoir pressure p_0. The resulting pressure wave travels towards the valve and fluid flows away from the reservoir at velocity v_0.

At $t = 4L/c$, the situation is the same as when the valve first closed at $t = 0$, and the cycle is repeated.

In practice, friction quickly dampens out the pressure waves and cavitation reduces the pressure decrease during the rarefactions.

A typical plot of pressure against time at a valve following rapid valve closure is superimposed on the theoretical plot in Figure 3.42.

3.16 GAS FLOW

General Relationships

The behavior of gases during processes involving thermal energy interactions and exchanges fits more properly into a study of thermodynamics. However, if only the flow mechanics are considered, the thermal and temperature effects may be restricted to those mainly relating to pressure and density. Applications of this approach are found in aircraft and rocket engines.

The most straightforward approach is to consider zero thermal energy transfer (heat transfer) to or from the fluid, or adiabatic flow. If, in addition, the changes in the fluid's properties are assumed to be reversible, then the flow becomes isentropic

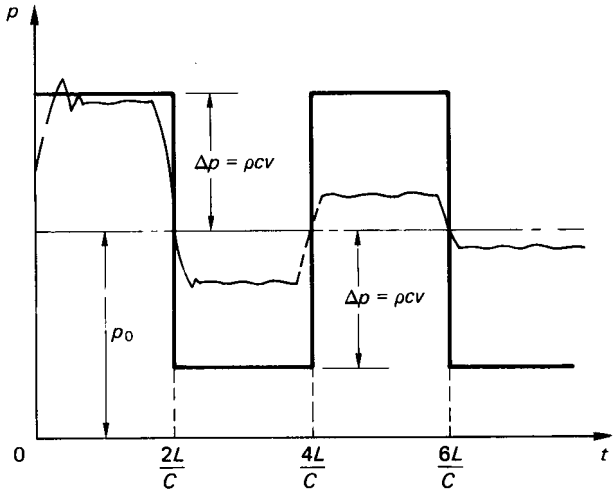


FIGURE 3.42 Pressure versus time at a valve.

and the relationship between pressure and density the simple and well-known one for an ideal gas:

$$\frac{p}{\rho^{\gamma}} = \text{constant} \tag{3.76}$$

Also, the *ideal gas law:*

$$\frac{p}{\rho} = RT \tag{3.77}$$

applies. Other useful relationships are:

1. The ratio of specific heats $\gamma = c_p/c_v$ (3.78)

2. The gas constant $R = c_p - c_v$ (3.79)

3. The *universal gas constant* $R_o = R \times$ molecular mass
$= 8.3143 \text{ kJ kg}_{\text{mol}}^{-1} \text{ K}^{-1}$

The terms *stagnation* or *total* temperature T_0 and pressure p_0 are often applied as the datum temperature and pressure of a fluid flow, even when stagnation conditions (zero velocity) do not exist in the particular situation under consideration. In gas flow the relationships between T_0 and the temperature T and p_0 and the pressure p at some point in the flow is often given in terms of the *Mach number* (M), the ratio of the flow velocity v to that of sound c, i.e.,

$$M = \frac{v}{c} \tag{3.80}$$

and

$$c = \left[\gamma \frac{p}{\rho}\right]^{0.5} = [\gamma RT]^{0.5} \tag{3.81}$$

In these terms T_0 and p_0 may be found from Euler's equation to be

$$T_0 = T\left[1 + \frac{(\gamma - 1)}{2} M^2\right] \tag{3.82}$$

$$p_0 = p\left[1 + \frac{(\gamma - 1)M^2}{2}\right]^{\gamma(\gamma-1)} \tag{3.83}$$

Flow in Ducts and Nozzles

Ducts. The analysis of gas flow in ducts is based on the Euler equation (3.26) and the one-dimensional continuity equation (3.18). Consideration of the differential forms of these equations will demonstrate that for subsonic flow ($M < 1$) the velocity will increase as the cross-sectional area of the duct decreases (in the converging entrance to a convergent/divergent nozzle, for example). For supersonic flow ($M > 1$) the velocity will increase as the cross-sectional area increases (in the diffuser of the convergent/divergent nozzle).

The properties of the fluid at a position in the flow stream where the local Mach number is unity are often denoted by a superscript $*$ (p^*, ρ^*, T^*) and used as a datum, so that

$$M^* = 1; \quad \text{and} \quad v^* = c^* = [\gamma RT^*]^{0.5} \tag{3.84}$$

The ratios of the properties at any position in the flow stream to those at the * position are:

$$1.\ \frac{p}{p^*} = \left[\frac{\gamma + 1}{2 + (\gamma - 1)M^2}\right]^{\gamma/(\gamma-1)} \tag{3.85}$$

$$2.\ \frac{\rho}{\rho^*} = \left[\frac{\gamma + 1}{2 + (\gamma - 1)M^2}\right]^{1/(\gamma-1)} \tag{3.86}$$

$$3.\ \frac{T}{T^*} = \frac{\gamma + 1}{2 + (\gamma - 1)M^2} \tag{3.87}$$

$$4.\ \frac{v}{v^*} = M\left[\frac{\gamma + 1}{2 + (\gamma - 1)M^2}\right]^{1/2} \tag{3.88}$$

$$5.\ \frac{A}{A^*} = \frac{1}{M}\left[\frac{2 + (\gamma - 1)M^2}{\gamma + 1}\right]^{(\gamma+1)/2(\gamma-1)} \tag{3.89}$$

For air the ratios may be calculated by substituting $\gamma = 1.4$, or obtained from published tables and charts (Houghton and Brock 1961).

Nozzles. A nozzle is an example of a duct with a smoothly decreasing cross-sectional area, followed in some cases by an increasing area (convergent/divergent nozzle) (see Figure 3.43). Since the velocity in the throat (minimum cross-section)

MECHANICAL ENGINEERING PRINCIPLES

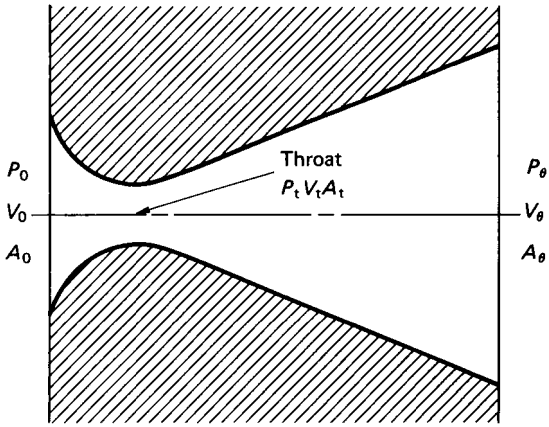


FIGURE 3.43 Convergent/divergent nozzle.

is often sonic, the approach velocity may be negligible. The throat velocity v_t and the exit velocity v_e are found by applying the Euler equation (3.26) between the upstream (entry conditions suffix$_0$) and throat and exit, respectively:

$$v_t = \left\{ \frac{2}{\gamma - 1} \frac{p_0}{\rho_0} \left[1 - \left(\frac{p_t}{p_0} \right)^{(\gamma-1)/\gamma} \right] \right\}^{1/2} \tag{3.90}$$

For v_e the pressure term p_e replaces p_t in equation (3.90). The mass flow rate through the nozzle is usually found at the throat by substituting v_t in the mass flow equation (3.19) to give

$$\dot{m} = C_d A_t \left\{ \frac{2}{\gamma - 1} \frac{p_0}{\rho_0} \left[\left(\frac{p_t}{p_0} \right)^{2/\gamma} - \left(\frac{p_t}{p_0} \right)^{(\gamma+1)/\gamma} \right] \right\}^{1/2} \tag{3.91}$$

where C_d is a discharge coefficient which depends on the nozzle design. For a well-designed nozzle C_d will be close to unity.

The mass flow rate will be the same at the exit as at the throat. It may be calculated from the exit conditions by substituting A_e and P_e for A_t and p_t, respectively, in equation (3.91).

Nozzles are usually designed for maximum mass flow rate. This will occur when the throat velocity is sonic ($v_t = c$). The pressure ratio which produces this situation is known as the *critical pressure ratio,* given by

$$\left(\frac{p_t}{p_0} \right)_{\text{crit}} = \left(\frac{2}{\gamma + 1} \right)^{\gamma/(\gamma-1)} \tag{3.92}$$

For many light diatomic gases such as air, where γ is approximately 1.4, $(pt_t)/(p_0)_{\text{crit}} = 0.528$.

The throat area will be that which gives the required mass flow rate through the throat at sonic velocity for critical pressure ratio. The exit area will be that which gives the calculated exit velocity for the given mass flow rate at the exit conditions. For convergent nozzles the throat also becomes the exit.

If nozzles, orifices, or Venturi meters are used to measure gas flow rates through a pipe then the approach velocity may be signficant and the mass flow rate given by

$$\dot{m} = C_d A_t \rho_0 \left\{ \frac{2(\rho_0/\rho_0)[\rho_t/\rho_0)^{2/r} - (\rho_t/\rho_0)^{(\gamma+1)/r}}{(\gamma - 1)[1 - (\rho_t/\rho_0)^{2/r}(\rho_t/\rho_0)^2} \right\}^{1/2} \tag{3.93}$$

Shock Waves

Under normal design conditions the flow in the nozzle downstream of the throat will be supersonic. The velocity of the gas at exit will depend on the external pressure p_b (back pressure). If the back pressure is greater than the theoretical exit pressure $p_e (p_b > p_e)$ then *shock waves* will be set up in the nozzle. These are discontinuities similar to standing waves in open-channel flow. The shock waves set up in such a way are *normal shock waves,* normal to the direction of flow.

If $p_b < p_e$ then the expansion will continue outside the nozzle (over-expansion).

If conditions upstream of a normal shock wave are denoted by suffix 1 and downstream by suffix 2, then it can be shown that the product of the up- and downstream velocities is equal to the square of the sonic velocity at $M = 1$:

$$v_1 v_2 = C^{*2} \tag{3.94}$$

and since $M_1 > 1$, then $M_2 < 1$. Also,

$$1.\ M_2 = \left[\frac{2 + (\gamma - 1)M_1^2}{2\gamma M_1^2 - (\gamma - 1)} \right]^{0.5} \tag{3.95}$$

$$2.\ \frac{p_2}{p_1} = \frac{2\gamma M_1^2}{(\gamma + 1)} - \frac{\gamma - 1}{(\gamma + 1)} \tag{3.96}$$

$$3.\ \frac{\rho_2}{\rho_1} = \frac{v_1}{v_2} = \frac{(\gamma + 1)M_1^2}{2 + (\gamma - 1)M_1^2} \tag{3.97}$$

These are known as the *Rankine–Hugoniot* relationships. Values for air may be obtained by putting $\gamma = 1.4$ or by the use of published tables (Houghton and Brock 1961).

The strength of a shock wave may be defined as the ratio of the pressure change across the wave to the upstream pressure, or in terms of the upstream Mach number:

$$\text{Shock wave strength} = \frac{p_2 - p_1}{p_1} = \left(\frac{2}{\gamma + 1} \right)(M_1^2 - 1) = 1.167(M_1^2 - 1), \text{ for air.} \tag{3.98}$$

Oblique shock waves, at an angle β to the upstream flow direction, are produced when a supersonic gas flow is turned through an angle θ by an obstruction such as an aircraft's nose, wing or tail, the inside walls of a duct, etc. The relationships between the up- and downstream Mach numbers and the angles β and θ are published in tables and charts (Houghton and Brock 1961).

In some cases both the up- and downstreams will be supersonic and subsequent shock waves produced, for example, at the leading and trailing edges of a wing. The effects of such shock waves produced by aircraft in flight, say, may be noted at ground level as *sonic booms*.

Gas Flow Measurement

Gas flow rates through ducts will normally be measured using devices and techniques similar to those used for incompressible fluids, namely orifice plates, venturi meters and nozzles. However, for gases the flow rate is usually quoted as a mass flow rate. Equations (3.90) and (3.92) may be used with orifices and Venturis as well as nozzles. Relevant values of C_d for each device will be found in BS 1042, in addition to operational advice.

When pitot-static tubes are used to measure gas flow velocities equation (3.49) may be acceptable for low flows with low pressure differences. At high velocities the compressibility must be taken into account and equation (3.90) used with stream conditions at X replacing those at the throat t.

For accurate velocity measurement with little disturbance to the flow *hot wire anemometers* may be used. The resistance of an electrically heated wire is related to the temperature of the wire, which in turn is related to the velocity of the fluid flow past the wire. The wire resistance measured on a bridge may be calibrated against a known velocity, to give either direct readout or (more usually) a calibration curve. The fine wire of the anemometer is suceptible to fluid contamination.

Other velometers and anemometers (depending on the relationship between the speed of rotation of a set of blades and the velocity (or speed) of the gas flow) may be used in very large cross-sectional ducts or to measure wind speed in the open air. They may depend on the rotation to generate a small electrical current, which can be calibrated as a speed, or the number of revolutions may be inserted into an empirical formula. A typical example is the three-vaned meteorological anemometer. Systems are now widely available for the measurement of velocity over a two dimensional field. Examples are laser doppler anemometry and particle image velocimetry, both of which require the seeding of the measured flow.

3.17 IDEAL FLUID FLOW

The concept of using idealized conditions to establish the shape of the mathematical models of real situations is common in engineering science studies. These models may then be modified to accommodate observed relationships, for application to real situations.

Ideal fluid (or *potential*) flow is such a concept. It may be used to set up flow patterns in the region of a flow stream outside the boundary layers. The combination of ideal flow and the boundary layer effects may be used to predict the performance of a real situation, so long as the limitations of both are recognized. The fluid is assumed to be inviscid and the flow steady, continuous, and irrotational. This means that there are no cavities or discontinuities in the flow stream, and that the fluid particles do not rotate about their own axes, even though the flow may be circular.

The continuity equation (3.16) applies, and may be modified to

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{3.99}$$

if required for two-dimensional flow. For irrotation in two dimensions equation (3.21) becomes

$$\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} = 0 \tag{3.100}$$

The Stream Function. From the definitions of the stream line and stream function the equation to a stream line may be shown to be

$$v_x dy - v_y dx = 0 \tag{3.101}$$

and since the stream function ψ is an equation which describes a family of stream lines, then, for example,

$$\psi = 2x - y$$

represents a family of parallel straight lines with a variable intercept ψ.

For unit thickness in the z direction the volumetric flow rate $\dot{V}$ between two stream lines 1 and 2 is

$$\dot{V} = \psi_1 - \psi_2 \tag{3.102}$$

or

$$d\dot{V} = d\psi \tag{3.103}$$

Stream functions may be *superposed* so that if

$$\psi_1 = \text{fn}(x,y) \text{ describes flow pattern } A$$

and

$$\psi_2 = \text{fn}(x,y) \text{ describes flow pattern } B$$

then $\psi = \psi_1 + \psi_2$ describes the flow pattern produced by the combination of A and B.

The x and y components of the flow velocity v are given by

$$v_x = -\frac{\partial \psi}{\partial y}; \quad \text{and} \quad v_y = \frac{\partial \psi}{\partial x} \tag{3.104}$$

or

$$v_x = \frac{\partial \psi}{\partial y}; \quad \text{and} \quad v_y = -\frac{\partial \psi}{\partial x}$$

depending on sign convention. In *polar coordinates* the components are

$$\text{Radial velocity } v_r = \frac{-\partial \psi}{r \partial \theta} \tag{3.105}$$

$$\text{Tangential velocity } v_\theta = \frac{\partial \psi}{\partial r} \tag{3.106}$$

The Velocity Potential

In a gravitational field there is a property the change in which is independent of the path of the change: *potential energy*. In a continuous, irrotational flow field there is also a property the change in which is independent of the path of the change. This property is the *velocity potential* (ϕ). It can be shown that

$$v_x = -\frac{\partial \phi}{\partial x}; \quad \text{and} \quad v_y = -\frac{\partial \phi}{\partial y} \tag{3.107}$$

or

$$v_\theta = -\frac{\partial \phi}{r \partial \theta}; \quad \text{and} \quad v_t = -\frac{\partial \phi}{\partial r} \tag{3.108}$$

Lines of constant ϕ are known as *velocity potential* lines with an equation

$$v_x dx + v_y dy = 0 \tag{3.109}$$

which intercept the stream lines at right angles to form an orthogonal network of characteristic pattern for each flow field.

From equations (3.104) and (3.107) it can be seen that

$$\frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x}; \quad \text{and} \quad \frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y} \tag{3.110}$$

which are the *Cauchy–Riemann* equations. In such flow fields the *Laplace* equations for ϕ and ψ must both be satisfied:

$$\nabla^2 \phi = 0; \quad \text{and} \quad \nabla^2 \psi = 0 \tag{3.111}$$

Because the fluid is ideal, the Bernoulli equation (3.27) may be readily applied between points in the field, both along and across the stream lines.

Flow Patterns

Examples of simple flow patterns are shown in Figure 3.44 with the equations to their stream functions and velocity potentials. A *source* is mathematically a point at which fluid appears and flows radially outwards. A *sink* is a negative source at which fluid flows radially inwards to disappear at a point (similar to the plug hole in a domestic sink, where, however, there is a vortex superposed to produce a spiral vortex or whirlpool). A *vortex* is flow in concentric circles with no radial flow. A *doublet* is the superposition of a source and a sink of equal strength m, initially a distance $2a$ apart brought infinitely close together so that the product of their strength and the distance between them remains a constant k. k is the strength of the doublet and is equal to $2am$.

This is a mathematical concept which is apparently impractical but yields a useful flow pattern. It is often used in combination with other simple patterns.

Modeling

Since there can be no flow across a stream line and the fluid is assumed inviscid, any stream line in a flow pattern may be replaced by a solid surface with no effect on the rest of the pattern. A stream line forming a closed contour may be replaced by a solid body to model the flow pattern around a body of the same shape. This provides a method of writing mathematical models to describe the flow streams around various shapes.

The limitations of the model must be understood, and the effects of rotation and viscosity particularly considered, when applying the analysis to *real* situations. For

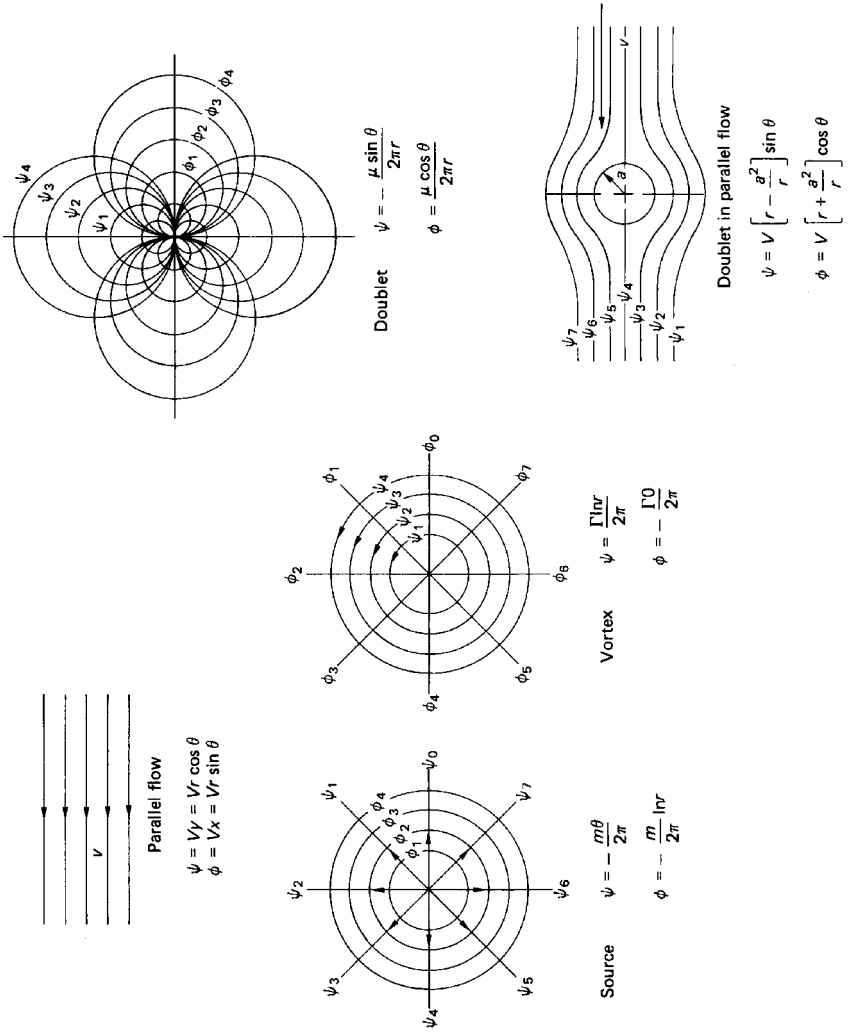


FIGURE 3.44 Simple flow patterns.

example, the flow round a cylinder may be modeled by a combination of doublet and parallel flow. The drag and lift forces calculated by integrating the resulting pressure forces at the cylinder surface appear to be zero. This is obviously not correct.

In the real situation the viscosity of the fluid produces a boundary layer at the cylinder surface, which, because of curvature, separates from the surface to form a wake. The presence of the wake disturbs the downstream flow pattern and the form drag force is a consequence. Viscosity also introduces a surface friction drag.

The shedding of vortices from the cylinder surface into the wake also produces alternate positive and negative lift forces, which are not predicted by the ideal flow analysis, although telegraph wires may often be observed vibrating in the wind.

3.18 CONCLUSION

Fluid mechanics is often regarded as an empirical subject which makes use of formulae based only on observed experimental results. This misconception is further compounded by the extensive use of coefficients (discharge especially) to account for effects which are difficult to model mathematically. However, almost all fluid mechanics equations in common use are based on the conservation of energy, the conservation of momentum or the fact that the rate of change of momentum may be equated to an applied force, usually a pressure force. The experimental checking and empirical amendment to derived formulae is just *good engineering practice.*

This section does not provide rigorous derivations of the various equations quoted. Some appreciation of such derivations may be required in order to establish the limitations and modifications necessary for the application of the equations, especially to nonstandard situations. For this, standard textbooks on fluid mechanics as listed below should be consulted.

PART 6

PRINCIPLES OF THERMODYNAMICS

Dennis H. Bacon

3.19 INTRODUCTION

Thermodynamics is concerned with energy transfers in processes. Two modes of transfer are recognized: *work* (transfer) and *heat* (transfer). Before proceeding, the terminology used in thermodynamics must be defined.

The particular part of the *working substance* under consideration is called the *system,* and this is separated from the *surroundings* by a *boundary.* In the *closed* or *nonflow* system the mass of working substance is constant, but in the *open* or *flow* system there is a mass flow rate across the boundary. Some processes in reciprocating plant may be considered by nonflow analysis but in steam plant, for example, most are considered by flow analysis. The *state* of a system is defined by the *properties* (pressure, temperature, etc.). Properties are normally expressed *specifically* (i.e., per unit mass) to enable charts or tables to be used. The state of simple substances can be described by two independent properties, but complex ones such as mixtures need more definition. A *change of state* is achieved by a *process* which is idealized as *reversible* with no losses. Reversible processes can be described by mathematical equations and enable analysis to be made to give answers for ideal situations. Real processes have losses and are described as *irreversible,* and the ideal results are multiplied by a coefficient or efficiency (based on measurement or experience) to predict real performance.

3.20 THE LAWS OF THERMODYNAMICS

The First Law of Thermodynamics

This is a law of energy conservation. When applied to a process we write

$$Q - W = \Delta E \quad \text{or} \quad q - w = \Delta e$$

where Q is the heat transfer (kJ) or q is the specific heat transfer (kJ/kg), W is the work transfer (kJ) or w is the specific work transfer (kJ/kg), and ΔE is the energy change (kJ) or Δe is the specific energy change (kJ/kg).

The change symbol Δ means final value minus initial value. ΔE embraces all forms of energy, but in the nonflow process it is usual to find that the only significant change is in the internal energy (U,u) and we write the nonflow energy equation

$$Q - W = \Delta U \quad \text{or} \quad q - w = \Delta u$$

For the *steady flow* system we write

MECHANICAL ENGINEERING PRINCIPLES

$$\dot{Q} - \dot{W}_x = \dot{m}\Delta\left(h + \frac{V^2}{2} + gz\right) \quad \text{or} \quad q - w_x = \Delta\left(h + \frac{V^2}{2} + gz\right)$$

where $\dot{Q}$ and $\dot{W}_x$ are the energy transfer rates and $\dot{m}$ is the steady mass flow rate across the boundary (in and out), Δh is the change in specific enthalpy ($h = u + pv$), $\Delta V^2/2$ is the change in specific kinetic energy, and Δgz is the change in specific potential energy. The suffix x is used on the work transfer to denote that this is the useful work from the system as the flow work is included in the enthalpy term. In flow problems it will also be necessary to use the continuity equation

$$\dot{m} = \rho A V$$

where ρ is the density and A is the area normal to the velocity V. Analysis of nonsteady flow may also be made, in which case energy terms to allow for the storage of energy in the system will be added.

Warning: A sign convention for work and heat is built into the equations above. Positive work means work obtained from the system and positive heat means heat put into the system. Care should be taken to be clear about the symbol V, which may appear as velocity or volume in many equations.

In order to allow continuous energy transfers a *cycle* is defined in which a series of processes brings the working substance back to the initial state so that the cycle can be repeated continuously. If we apply the first law to a cycle it follows that ΔE is zero and

$$\sum_{\text{cycle}} Q = \sum_{\text{cycle}} W$$

The Second Law of Thermodynamics

It might be thought that the first law of thermodynamics permits all the heat transfer to a cycle to be returned as work transfer, but unfortunately the second law places restraints on the achievement of this desirable situation. The restraint takes the practical form of demanding that some of the heat transfer to the cycle *must* be rejected as a heat transfer to a lower temperature. Thus when we build a *heat engine* it has to exchange heat with (at least) two reservoirs in order to produce work (Figure 3.45). Since work is the objective, the amount produced per unit heat input is vital information and we define the *thermal efficiency* of a heat engine as

$$\eta_{\text{thermal}} = \frac{\text{Net work transfer from the cycle}}{\text{Heat transfer to the cycle}} = \frac{W}{Q_1}$$

Since the first law states $Q_1 - Q_2 = W$ we see that efficiency is less than unity.

The second law makes further investigations and determines the maximum possible efficiency of a heat engine using reversible isothermal processes to transfer heat from two reservoirs as

$$\eta_{\text{thermal maximum}} = 1 - (T_{\min}/T_{\max})$$

This efficiency is known as the Carnot efficiency and is not attainable due to losses. It is also found that constant temperature processes, except during phase change, are not practical and real processes of heat transfer take place at approximately

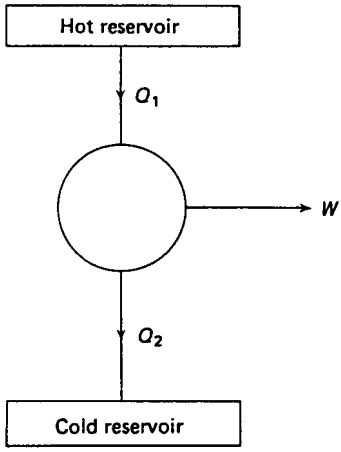


FIGURE 3.45 A heat engine.

constant volume or constant pressure. One positive product of the second law is that it tells the engineer that thermal efficiency will be increased by making the maximum cycle temperature as high as possible (a materials constraint) and by making the minimum cycle temperature as low as possible (ambient conditions).

The losses in a real cycle are due to internal fluid friction and the necessity of having a temperature difference to cause a heat transfer. The larger the temperature difference, the greater the losses. The fluid friction losses in a work-producing process are defined by the process efficiency

$$\eta_{\text{process}} = \frac{\text{Actual work produced}}{\text{Ideal work produced}}$$

which is inverted for work-absorbing processes (compression). The most common application of this efficiency is in steady flow *adiabatic* processes. Ideally, these are processes with no heat transfer which are often used as models for real processes in which the heat transfers are negligible compared with the work transfers (turbine expansion). In such processes the efficiency is known as the isentropic efficiency because the ideal adiabatic process has constant entropy.

Losses due to fluid friction and losses due to heat transfer across finite temperature differences are found to result in an increase in the value of the entropy that would be expected in a reversible process. Thus, an expected increase would be larger and an expected decrease would be smaller. It is not easy to define entropy except mathematically. In practical use as the abscissa of charts it enables work transfers in ideal adiabatic processes to be represented as vertical lines if enthalpy is used as an ordinate, and in this guise is a valuable visual method of presentation.

3.21 THERMOECONOMICS

When a more detailed study, see Kotas (1985) and Bejan (1982), of a flow process is made by the second law of thermodynamics it is found that specific entropy (J/

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kgK) appears as part of a property known as availability. In a flow process we write $b = h - T_0 s$, in which b is the specific *availability* function, h is the specific enthalpy, T_0 is the temperature (absolute) of the surroundings, and s is the specific entropy. The second law shows that the maximum work potential or *exergy* of any state in surroundings at state 0 is given by $b - b_0$. Thus for a change of state in a flow process from 1 to 2 the maximum specific work obtainable is given by the exergy change, $w_{x_{max}} = (b_1 - b_0) - (b_2 - b_0) = (b_1 - b_2) = -\Delta b$. If we measure or predict by analysis the actual work achieved it is possible to determine numerically the lost work or irreversibility in the process. If engineering plant is to be designed to the best advantage it is clear that processes should be chosen to minimize this loss. The lost work may be associated with costs and we move into the developing field of thermoeconomics. Clearly, this is a complex subject but it is important in that it unites thermodynamics with costs and can help in the design of long-life expensive plant, such as aircraft engines.

3.22 WORK, HEAT, PROPERTY VALUES, PROCESS LAWS AND COMBUSTION

To deploy the laws of thermodynamics outlined above we need more information. To perform simple cycle analysis the data below are vital.

Work

In a nonflow process work transfer can be determined from $w = \int p dv$. The mathematical relation for the process is known as the process law (qv). In most flow processes used in engineering cycles the adiabatic approximation is used so that the steady flow energy equation, neglecting changes in kinetic and potential energy, gives

$$w_x = \Delta h$$

Heat

This is usually an unknown quantity and is found by the application of the energy equation. As stated earlier, many processes are approximately adiabatic so that heat transfer is zero and in others heat transfer is obtained from combustion data or, if a heat exchange process, by heat exchanger efficiency.

Property Values

These are found in tables or from charts for common substances. Computer formulations are now widely available.

Process Laws

This is a particularly important step in thermodynamic analysis because an idealized reversible process has to be chosen to represent as closely as possible the real

process in order to calculate energy changes. When the working substance is a gas, it is convenient in an elementary analysis to use perfect gas laws with the process calculation. These are

$$pv = RT \quad \text{or} \quad pV = mRT$$

$$u = c_v \Delta T \quad \text{and} \quad h = c_p \Delta T$$

where c_v and c_p are the specific heat capacities at constant volume and constant pressure, respectively, which are related as follows:

$$c_p - c_v = R \quad \text{and} \quad c_p / c_v = \gamma$$

where R is the specific gas constant and γ is the isentropic index.

Ideal processes commonly used are *constant pressure, constant volume, constant temperature* (which for a perfect gas becomes pv = constant) together with two other more general relations: the *adiabatic process,* pv^k = constant (which for a perfect gas becomes pv^γ = constant), and the *polytropic process,* pv^n = constant. The last process is a general relation between pressure and volume which is used if none of the other clearly special cases are considered valid. Usually $1 < n < 1.4$.

It is possible (by using the gas laws) in adiabatic and polytropic gas processes to rearrange the relations to involve pressure and temperature or temperature and volume to yield very useful relations:

$$\frac{T_1}{T_2} = \left(\frac{p_1}{p_2}\right)^{(\gamma-1)/\gamma}; \frac{T_1}{T_2} = \left(\frac{V_2}{V_1}\right)^{\gamma-1}: \frac{T_1}{T_2} = \left(\frac{p_1}{p_2}\right)^{(n-1)/n}; \frac{T_1}{T_2} = \left(\frac{V_2}{V_1}\right)^{n-1}$$

Processes may be represented on property diagrams to enable cycle visualization (Figures 3.46–3.51).

Combustion

To avoid involving complex chemical equations, engineers often use the calorific value of a fuel coupled with a combustion efficiency to estimate the energy transfers in combustion processes. Thus, the rate of energy input by combustion is

$$\dot{E} = \dot{m}_f \cdot CV \cdot \eta_{\text{comb}}$$

where $\dot{m}_f$ is the fuel mass flow rate, CV the calorific value of the fuel, and η_{comb} the combustion efficiency.

3.23 CYCLE ANALYSIS

One example will be given of the simple analysis of the ideal Joule cycle for a gas turbine plant (Figure 3.51). The cycle consists of four flow processes described in Table 3.6 and analyzed by the steady flow energy equation.

From the data in the table it can be seen that the specific work $w = c_p(T_3 - T_4) - c_p(T_2 - T_1)$ and the thermal efficiency

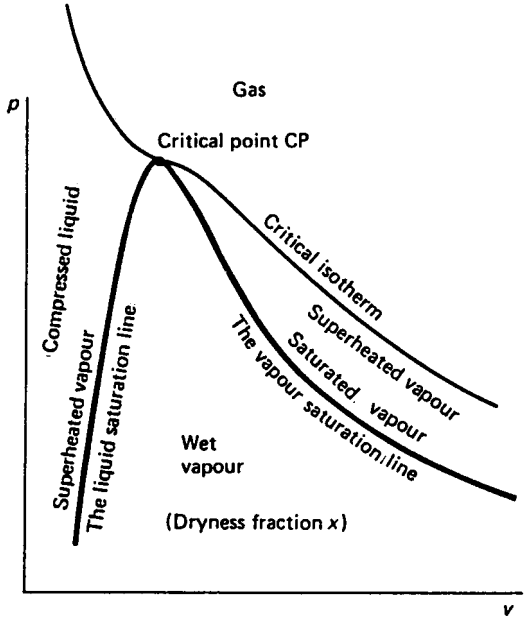


FIGURE 3.46 Substance phases and definitions.

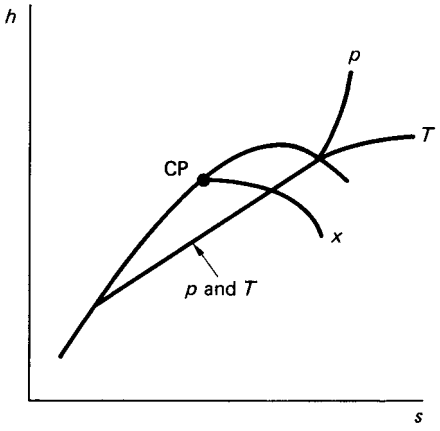


FIGURE 3.47 Steam processes on an *h–s* diagram.

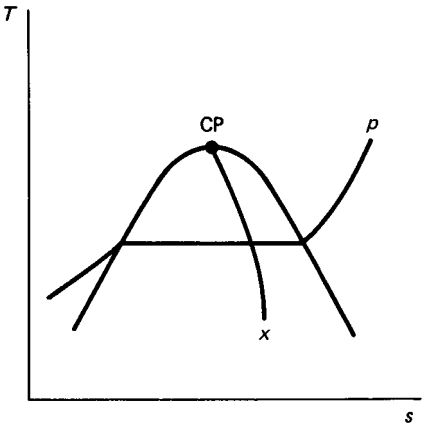


FIGURE 3.48 Steam processes on a T–s diagram.

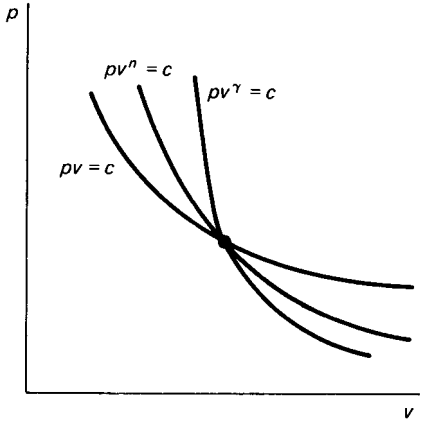


FIGURE 3.49 Gas processes on a p–v diagram.

$$\eta_{\text{thermal}} = \frac{c_p(T_3 - T_4) - c_p(T_2 - T_1)}{c_p(T_3 - T_2)}$$

If allowance is made for the isentropic efficiency of the compression and expansion processes the cycle diagram is changed to show the associated entropy increases but the expressions for work and efficiency above are still valid with the changed values of T_2 and T_4 (Figure 3.52). These values are determined from the use of the reversible adiabatic process relation and the isentropic efficiency as

$$T_2 - T_1 = T_1(r_p^{(\gamma-1)/\gamma} - 1)/\eta_c$$

and

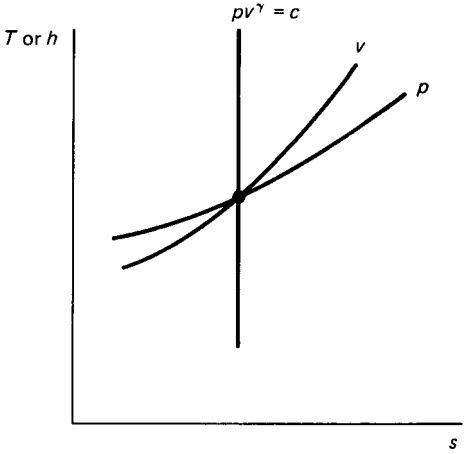


FIGURE 3.50 Gas processes on a T–s diagram.

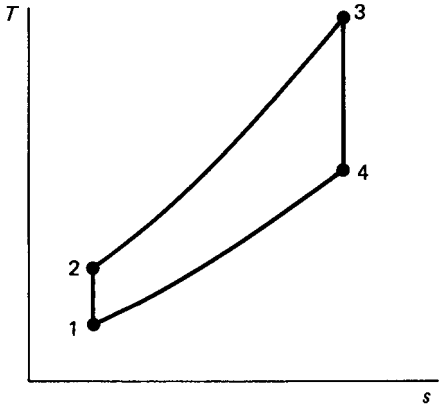


FIGURE 3.51 The Joule cycle.

$$T_3 - T_4 = \eta_t T_3 (1 - 1/r_p^{(\gamma-1/\gamma)})$$

where r_p is the cycle pressure ratio,

$$\eta_c = \frac{T_2' - T_1}{T_2 - T_1}$$

and

$$\eta_t = \frac{T_3 - T_4}{T_3 - T_4'}$$

η_c and η_t being the isentropic efficiencies of compression and expansion. If these values are substituted into the work and thermal efficiency expressions they become

TABLE 3.6

Process	Description	$q - w_x = h$
1 to 2	Reversible adiabatic compression $pv^\gamma = \text{constant}$	$-w_x = h_z - h_1 = c_p(T_2 - T_1)$
2 to 3	Reversible constant pressure heat transfer *to* the cycle	$q = h_3 - h_2 = c[p(T_3 - T_2)$
3 to 4	Reversible adiabatic expansion $pv^\gamma = \text{constant}$	$-w_x = h_4 - h_3\ w_x = c_p(T_3 - T_4)$
4 to 1	Reversible constant pressure heat transfer *from* the cycle	$q = h_1 - h_4 = c_p(T_1 - T_4)$

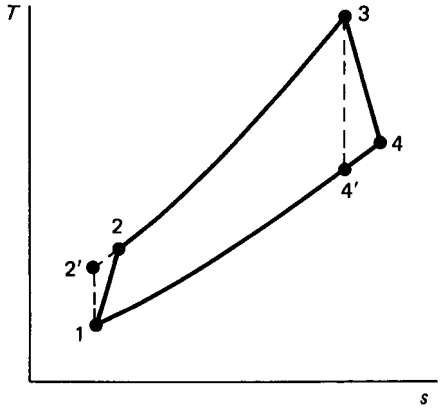


FIGURE 3.52 The effect of isentropic process efficiency on the Joule cycle.

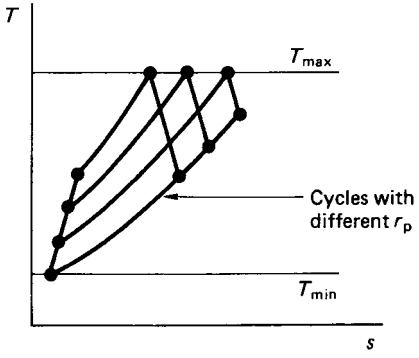


FIGURE 3.53 The effect of pressure ratio in a cycle with fixed T_{max} and T_{min}.

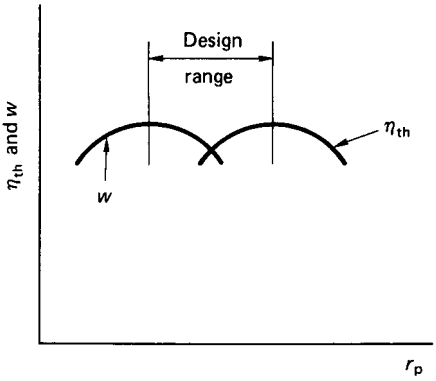


FIGURE 3.54 Thermal efficiency and specific work transfer variation in a Joule cycle with allowance for isentropic process efficiency.

more useful in that they involve the thermodynamically significant maximum and minimum cycle temperatures which are fixed by material and ambient conditions respectively, so that the only variable is the cycle pressure ratio (Figure 3.53). If the expressions are differentiated with respect to this pressure ratio it is possible to find the pressure ratio for maximum work and that for maximum efficiency. The cycle designer then has a choice, depending on the proposed application and Figure 3.54 shows that it would be expected that the chosen ratio would fall between these two maxima. Obviously, this simple approach is not the complete answer to gas turbine cycle analysis, but it illustrates the use of the laws of thermodynamics, and similar work may be done for other plant cycles.

PART 7
HEAT TRANSFER

Dennis H. Bacon

3.24 INTRODUCTION

Whenever a temperature difference occurs there is an energy flow from the higher temperature to the lower. A study of heat transfer is concerned with the determination of the instananeous rates of energy flow in all situations. We determine heat transfer rates in watts. These rates will be constant in situations where the temperature difference remains constant but variable (transient) when the temperature difference varies either due to the heat transfers or to other energy changes such as internal chemical reaction.

There are three modes of heat transfer:

1. *Conduction,* which is of greatest interest in solid bodies but also occurs in fluids, where it is often overshadowed by convection
2. *Convection,* which occurs in fluids when energy is transferred due to the motion of the fluid
3. *Radiation,* which occurs between two systems at different temperatures which need not be in contact provided any intervening medium is transparent to the radiation.

In practice, all three modes may occur simultaneously and it is necessary to draw up a balance at a boundary. For example, energy may be conducted to the surface of an electric storage heater and is then convected and radiated to the surroundings. Thus calculations can become complex, and in this particular case where energy is added at certain times this is a continuously varying situation.

Three approaches to heat transfer will be discussed below:

1. A simple method suitable for many estimations
2. A more detailed appraisal of the field
3. Comments on the use of computers

3.25 BASIC PRINCIPLES OF HEAT TRANSFER (*White 1984; Özişik 1985; Kreith and Bohn 1986*)

Conduction

Fourier's law for conduction states:

$$q = -k \frac{dT}{dx}$$

The thermal conductivity $k(\mathrm{Wm^{-1}K^{-1}})$ is a property of the material which varies

with temperature but for small temperature ranges is usually considered constant. Typical values are shown in Table 3.7. With constant k Fourier's equation can be integrated for four common situations.

Plane Surfaces. Integration gives

$$\dot{Q} = k_{12}A\frac{(T_1 - T_2)}{x_2 - x_1} = \Delta T/\left(\frac{\Delta x}{Ak}\right)_{12}$$

(see Figure 3.55). The quantity $(\Delta x/kA)$ is known as the thermal resistance in KW^{-1}. Thermal resistances can be added in a similar way to electrical resistances so that for a multilayer plane surface there are a number of resistances in series (Figure 3.56). Thus we can write

$$Q = (T_1 - T_4)/\sum\frac{\Delta x}{kA}$$

Cylindrical Surfaces. For tubes it is more convenient to evaluate heat transfer rates per unit length, and integration gives

$$\dot{Q} = (T_1 - T_2)/\frac{\ln r_2/r_1}{2\pi k_{12}L} = \Delta T/\frac{\ln r_2/r_1}{2\pi k_{12}L}$$

(see Figure 3.57), and in this case the thermal resistance is

TABLE 3.7

Substance	Thermal conductivity, $Wm^{-1}\ K^{-1}$, at 20°C
Aluminum	204
Iron	52
Water	0.597
Air	0.026 (100 kPa)
Glass wool	0.04

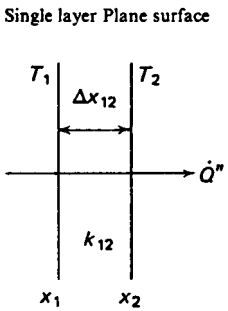


FIGURE 3.55 One-dimensional conduction through a single-layer plane wall.

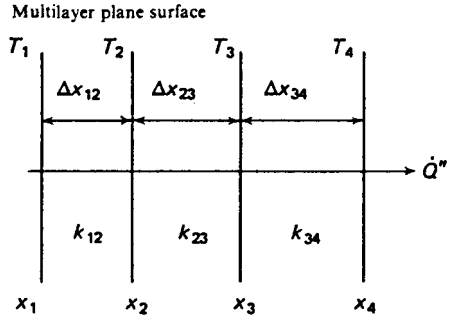


FIGURE 3.56 One-dimensional conduction through a multilayer plane.

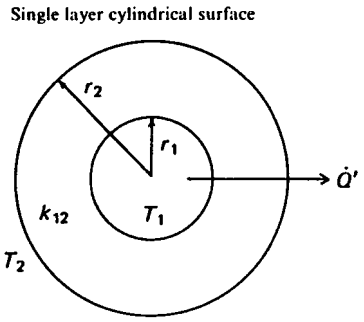
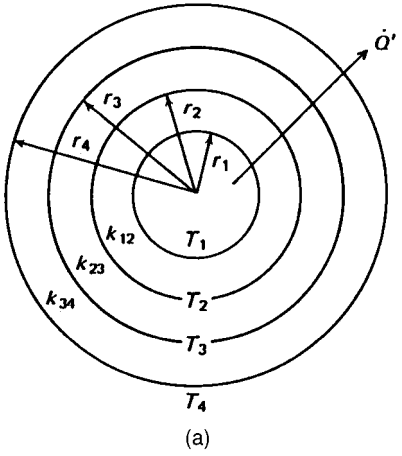


FIGURE 3.57 Conduction through a single-layer cylindrical surface.



(a)

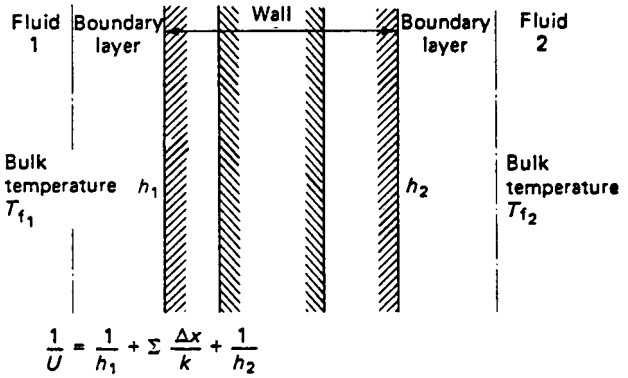
FIGURE 3.58a Conduction through a multilayer cylindrical surface; *U*-value for a plane surface.

$$\left(\frac{\ln r_2/r_1}{2\pi k_{12}L}\right) \text{ in KW}^{-1}$$

For a multilayer tube, thermal resistances are added to give

$$\dot{Q}' = (T_1 - T_4)/\sum\left(\frac{\ln (r_{\text{outer}}/r_{\text{inner}})}{2\pi kL}\right)$$

(see Figure 3.58).



(b)

FIGURE 3.58b *U*-value for a cylindrical surface.

Convection

The fundamental equation for convective heat transfer at a solid–fluid interface is

$$\dot{Q}'' = h\theta$$

where θ is the temperature difference between surface and fluid. The surface heat transfer coefficient $h(\mathrm{Wm^{-2}K^{-1}})$ is not a property of the fluid or the surface but depends on the flow rate, the fluid properties, and the surface shape. The coefficient has to be determined for each situation and can vary considerably (Table 3.8). Although the determination of h is crucial to convection calculations, it is an extremely difficult process, and accurate prediction of convective heat transfer is not always possible.

If we express the convection equation in a thermal resistance form suitable for plane surfaces,

$$\dot{Q} = \frac{\theta}{nA}$$

it can be seen that the thermal resistance is $(2/nA)\mathrm{KW^{-1}}$. For tubular surfaces it is again more convenient to work per unit length, so that

$$\dot{Q}' = \theta \Big/ \left(\frac{1}{2\pi rh}\right)$$

and the thermal resistance is $(1/2\pi rh)\mathrm{mKW^{-1}}$.

Overall Heat Transfer Coefficients

A common heat transfer situation is a solid wall separating two fluids, and for this problem the thermal resistances for conduction and convection can be added to enable the heat transfer rate to be determined in terms of the two fluid temperatures.

For a plane surface (Figure 3.58(b),

$$\dot{Q}'' = (T_{f_1} - T_{f_2}) \Big/ \left(\frac{1}{h_1} + \sum \frac{\Delta x}{k} + \frac{1}{h_2}\right)$$

For a tubular surface (Figure 3.59),

TABLE 3.8 Range of values of surface heat transfer coefficient ($\mathrm{W\ m^{-2}K^{-1}}$)

Free convection	Gases	0.5 to 500
	Liquids	50 to 2000
Forced convection	Gases	10 to 700
	Liquids	100 to 10 100

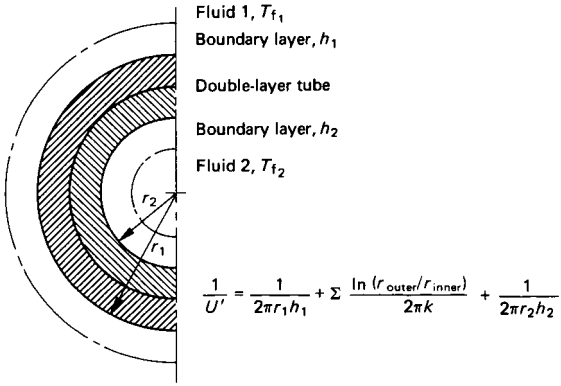


FIGURE 3.59 U'-value for a cylindrical surface.

$$\dot{Q}' = (T_{f_1} - T_{f_2}) \Big/ \left(\frac{1}{2\pi r_1 h_1} \right) + \sum \left(\frac{\ln(r_{\text{outer}}/r_{\text{inner}})}{2\pi k} \right) + \frac{1}{2\pi r_2 h_2}$$

It can be seen that the added resistances may be inverted to give an overall *conductance* which is known as a *U*-value or overall heat transfer coefficient.

For a plane surface,

$$\frac{1}{U} = \frac{1}{h_1} + \sum \frac{\Delta x}{k} + \frac{1}{h_2}$$

For a tubular surface,

$$\frac{1}{U'} = \frac{1}{2\pi r_1 h_1} + \sum \left(\frac{\ln(r_{\text{outer}}/r_{\text{inner}})}{2\pi k} \right) + \frac{1}{2\pi r_2 h_2}$$

The heat transfer rate is then simply written

For a plane surface,

$$\dot{Q} = UA(T_{f_1} - T_{f_2}) \text{ where } A \text{ is the area}$$

For a tubular surface

$$\dot{Q} = U'l(T_{f_1} - T_{f_2})$$

where l is the length.

The situation in which this technique is commonly used is in heat exchanger design. It should suffice for simple calculations provided suitable values of the surface heat transfer coefficients for convection can be obtained. (*CIBS Guide* 1980). If the temperature difference is not constant then a mean value should be used. A suitable equation for a mean can be found in the heat exchanger section which follows.

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Radiation

Radiation is of central importance in space application where all rejected heat is radiated to space. The rate of energy emitted by an ideal black body is given by the Stefan–Boltzmann law, in which the absolute temperature (Celsius + 273) is raised to the fourth power:

$$\dot{E}''_b = \sigma T^4$$

where σ is the Stefan–Boltzmann constant $5.67 \times 10^{-8}\text{Wm}^{-2}\text{K}^{-4}$ and the subscript b refers to the ideal black body. Real bodies emit less radiation, and the monochromatic emissivity is defined by

$$\varepsilon_\lambda = \left[\frac{\dot{E}''_\lambda}{\dot{E}''_{b\lambda}}\right]_T$$

The value of ε varies with λ and T because real bodies are selective emitters, but for simple calculations it is often assumed that emissivity is constant. The calculations associated with this assumption are based on grey body theory, for which the rate of energy emission is given by

$$\dot{E}''_g = \varepsilon\sigma T^4$$

It would be unwise to estimate unknown emissivities, and measurements would need to be made unless suitable data could be found.

Radiation incident on a body may be absorbed, reflected or transmitted. Thus we write $\alpha + \rho + \tau = 1$ where α, ρ and τ are the absorptivity, reflectivity, and transmissivity, respectively. Ideal black bodies absorb all incident radiation but real bodies do not. Gases are often assumed to transmit all radiation, but this is not always true, particularly with hydrocarbon combustion products and atmospheric transmission. Solids have a transmissivity of zero. With these simple ideas it is necessary to know the values of only α and ρ. It can be shown that a gray body has absorbtivity equal to emissivity, $\alpha = \varepsilon$. Thus, provided the transmissivity is zero, a knowledge of the grey body emissivity enables reflectivity to be determined, since

$$\rho = 1 = \varepsilon$$

The only simple radiation problem that can be solved with the simple approach above is that of a gray body in large surroundings (see Figure 3.60). The word 'large' implies that radiation not incident on the grey body which will be incident on the surroundings and will therefore be reflected will not be re-incident on the gray body. Thus, the surroundings are effectively black. (This might be true with a linear size factor greater than 10.) In this simple case it can be shown that the heat transfer rate is

$$\dot{Q} = \varepsilon\sigma A(T^4 - T_s^4)$$

where ε is the emissivity of the body, A is the area of the body, T is the temperature of the body in K and T_s is the temperature of the large (black) surroundings in K.

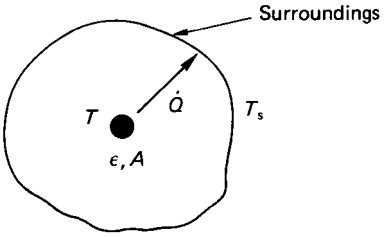

FIGURE 3.60 A small gray body in large (black) surroundings.

Simple Transient Problems

If a body is being cooled or heated by convection or radiation and the thermal conductivity is large so that the rapid heat transfer rates within the body enable it to be assumed that the body temperature distribution is uniform, then the situation is known as a lumped capacity system. For such a system the complex methods of transient heat transfer are not required and a simple energy balance equation may be drawn up and integrated. The most common case is quenching, a convective boundary problem for which in time dt a small heat transfer δQ occurs when the body temperature changes from T by an amount dT (Figure 1.61). Thus,

$$\delta Q = \rho c_p V dT = -hA(T - T_f)$$

where T_f is the fluid temperature, A is the body surface area, V is the body volume, ρ is the body density, c_p is the body specific heat, and h is the surface heat transfer coefficient. Integration gives

$$\frac{\theta}{\theta_0} = e^{-(hAt/\rho c_p V)}$$

where θ_0 is the initial temperature difference between fluid and body and θ is the temperature difference at any future time t. The quantity $(\rho c_p V/hA)$ may be regarded as the time constant of the system.

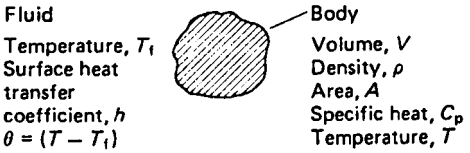

FIGURE 3.61 A lumped-capacity system with convection.

3.26 ANALYSIS OF HEAT TRANSFER

Conduction

By considering the thermal equilibrium of a small, three-dimensional element of solid, isotropic material it can be shown that for a rectangular coordinate system

$$\frac{\partial T}{\partial t} = \alpha \left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right] + \frac{\dot{Q}'''}{\rho c_p}$$

where $\partial T/\partial t$ is the rate of change of temperature with time, α is the thermal diffusivity of the material $\alpha = k/\rho c_p$ and $\dot{Q}'''$ is the internal heat generation rate per unit volume, which may be due, for example, to electric current flow for which $\dot{Q}''' = i^2 r$, where i is the current density and r the resistivity. The solution to this equation is not easy, and numerical approximation methods are often used. One such method is the finite difference technique, in which continuously varying temperatures are assumed to change in finite steps. Consider the three planes shown a distance Δx apart (Figure 3.62). At

$$A \frac{\partial T}{\partial x} = \frac{T_1 - T_2}{\Delta x}$$

and at

$$B \frac{\partial T}{\partial x} = \frac{T_2 - T_3}{\Delta x}$$

so that at

$$C \frac{\partial^2 T}{\partial x^2} = \frac{T_1 + T_3 - 2T_2}{\Delta x^2}$$

Similarly, $\partial T/\partial t$ may be written

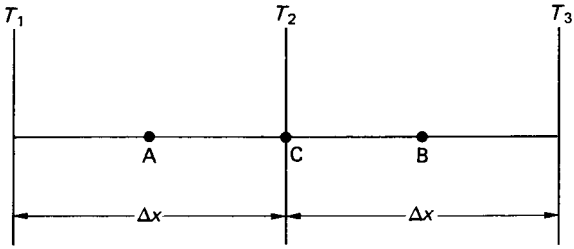

FIGURE 3.62 One-dimensional finite difference formulation.

$$\frac{T_{n,1} - T_{n,0}}{\Delta t}$$

where $T_{n,1}$ is the temperature at layer n at time 1 and $T_{n,0}$ is the temperature at layer n at time 0.

For steady state situations $\partial T/\partial t = 0$ and a two-dimensional plane surface will be used for illustration requiring a solution of

$$\alpha \left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right] + \frac{\dot{Q}'''}{\rho c_p} = 0$$

Consider the surface to be divided by a grid (Figure 3.63). It is then found that the solution for any point in the plane for steady state conduction without heat generation is

$$T_1 + T_2 + T_3 + T_4 - 4T_0 = 0$$

or with heat generation is

$$T_1 + T_2 + T_3 + T_4 - 4T_0 = -\frac{a^2 \dot{Q}'''}{k}$$

At the boundary of the plane conditions are usually isothermal, in which case $T =$ constant, or convective when an energy balance yields for a straight boundary (Figure 3.64)

$$\frac{T_1}{2} + T_2 + \frac{T_3}{2} + \frac{ha}{k} \cdot T_f - T_0 \left(2 + \frac{ha}{k} \right) = 0$$

where h is the surface heat transfer coefficient, T_f is the fluid temperature, and a is the grid size. Similar expressions can be derived for corners, curves, etc. at the boundary.

For a large number of grid points, a large number of simultaneous equations are obtained which can be solved by iteration or Gaussian elimination. Computer programs may be used to advantage. The solution obtained will be the temperature

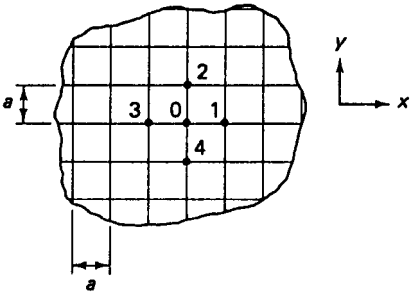

FIGURE 3.63 The two-dimensional grid concept.

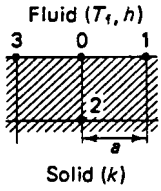

FIGURE 3.64 Convective surface nomenclature.

distribution in the plane, and the heat transfer rates may be found at the boundary (Figure 3.65):

With an isothermal boundary $\dot{Q} = \Sigma k(T_m - T_{wall})$

With a convective boundary $\dot{Q} = \Sigma ha(T_f - T_m)$

For transient heating or cooling for which $\partial T/\partial t \neq 0$, a one-dimensional illustration is used. The equation to be solved is

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

when there is no heat generation (Figure 3.66).expressed in finite difference form this becomes

$$T_{n,1} = F\left(T_{n-1,0} + T_{n+1,0} + T_{n,0}\left(\frac{1}{F} - 2\right)\right)$$

where F is the nondimensional grid size Fourier number $F = \alpha \Delta t/a^2$. The only unknown in this equation is $T_{n,1}$, the temperature at layer n after one time interval Δt. Thus from a knowledge of the initial conditions successive temperatures in each layer can be found directly for each time interval. This is the *explicit* method and is used for tabular or graphical (Schmidt method) solutions. If $F > 0.5$ the solution

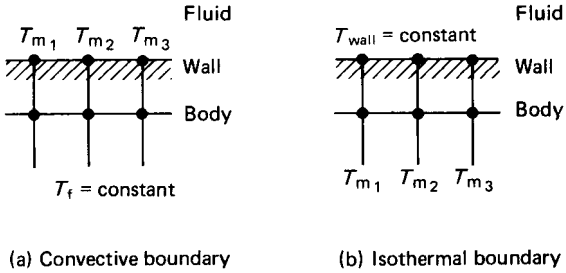

FIGURE 3.65 Calculation of heat transfer rate.

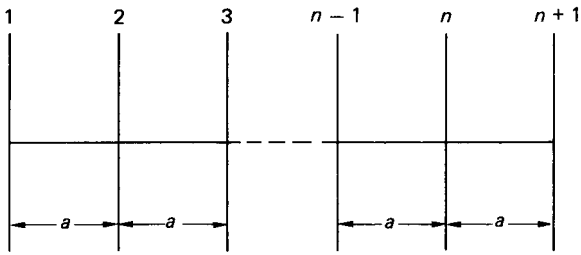

FIGURE 3.66 One-dimensional transient conduction formulation.

is unstable and in three dimensions the criterion becomes severe. The boundary conditions may be isothermal or convective and in the latter case the solution is

$$T_{n,1} = 2F \cdot T_{n-1,0} + T_{n,0}[1 - 2F - 2F \cdot B] + 2F \cdot BT_f$$

where B is the nondimensional grid Biot number, $B = ha/k$. For this case the solution is unstable if $(F + FB) > 0.5$. The solutions obtained give the temperature distribution in the one-dimensional plane and the heat transfer is found at the boundary

$$Q = \sum_t kA \left(\frac{\partial T}{\partial x}\right)_{1,t} \Delta t$$

or from the temperature profile

$$Q = \sum_{\text{layer}} mc_p(T_{\text{final}} - T_{\text{initial}})$$

The stability problems of the explicit method can be overcome by the use of *implicit* methods for which there is no direct solution, but a set of simultaneous equations are obtained which may be solved by Gaussian elimination. A computer program may be used to advantage. A satisfactory implicit method is that due to Crank and Nicolson. The importance of a stable solution is that if the choice of F is limited then the grid size and time interval cannot be freely selected, leading to excessive calculations for solution. The implicit method releases this constraint but care is still needed to ensure accuracy.

Although the finite difference method has been chosen for demonstration because the method is easy to understand, most modern computer programs are based on the finite element technique. However, the mathematical principles are involved, and would not lend themselves to simple programming. Before the availability of computer software analytical solutions were obtained and presented as graphs of transient solutions for slabs, cylinders, and spheres. These graphs enable solutions for other shapes to be obtained by superposition methods. Such methods should be used to avoid or validate computer solutions.

Warning: If fiber-reinforced materials are used in which the lay-up is arranged to give directional structural strength it will be found that the thermal conductivity has directional variation and the methods above will need considerable amendment.

Convection

A knowledge of the surface heat transfer coefficient h is essential in determining heat transfer rates. Fluid flow over a solid surface is a boundary layer problem, and the heat transfer depends on boundary layer analysis. This analysis may be by differential or integral approach, but solution is difficult and the modeling of turbulence is complex. Computer solutions based on numerical approximations may be used to advantage, but simple approaches have been used for many years and are still extremely useful. These methods are based on Reynolds' analogy (modified by later workers) and dimensional analysis backed by experimentation.

Convection may be free or forced. In *forced convection* it is found that the heat transfer coefficient can be included in a nondimensional relation of the form

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$$Nu = \phi(Re,Pr) = \text{constant} \cdot Re^a \cdot Pr^b$$

where Nu is the Nusselt number ($Nu = hl/k$), Re is the Reynolds number ($Re = \rho Vl/\mu$), and Pr is the Prandtl number ($Pr = \mu c_p/k$). In these relations l is a representative length dimension (diameter for a pipe and some chosen length for a plate), V is the bulk or free stream velocity outside the boundary layer. The values of the constants a and b depend on whether the flow is laminar or turbulent and on the geometry of the situation, and are usually found by experiment.

The determination of whether flow is laminar or turbulent is by the value of the Reynolds number;

For plates, $Re < 500\,000$, flow is laminar: $Re > 500\,000$, flow is turbulent

For tubes, $Re < 2000$, flow is laminar: $Re > 4000$, flow is turbulent

(between these two values there is a transition zone). There are many relations to be found in texts which allow for entry length problems, boundary conditions, etc. and it is not feasible to list them all here. Two relations are given below which give average values of Nusselt number over a finite length of plate or tube in forced, turbulent flow with Mach number less than 0.3 using total plate length and diameter for representative length dimension. Care must be taken in any empirical relation to use it as the author intended.

Plate: $$Nu = 0.036Re^{0.8}Pr^{0.33}$$

In this relation fluid properties should be evaluated at the film temperature, $T_{\text{film}} = (T_{\text{wall}} + T_{\text{bulk}})/2$.

Tube: $$Nu = 0.023Re^{0.8}Pr^{0.4}$$

In this relation fluid properties should be evaluated at the bulk temperature, $0.6 < Pr < 160$ and $(l/d) > 60$.

It should be noted that the index of Reynolds number of 0.8 is characteristic of turbulent flow; in laminar flow 0.5 is found.

It must be emphasized that reference to other texts in all but these simple cases is essential to estimate heat transfer coefficients. It should also be pointed out that the values obtained from such relations could give errors of 25%, and a search of the literature might reveal equations more suited to a particular situation. However, an estimate within 25% is better than no knowledge, and is a suitable starting point which may be modified in the light of experience.

For complex heat exchange surfaces such as turbine blade cooling in an aircraft engine, empirical information is usually presented graphically (on these graphs the nondimensional group St (Stanton number) may appear:

$$St = \frac{Nu}{RePr} = \frac{h}{\rho Vc_p}$$

In *free convection* the relationship used is $Nu = \phi(Pr \cdot Gr)$, where Gr is the Grashof number, $\rho^2\beta g\theta l^3/\mu^2$ in which β is the coefficient of cubical expansion of the fluid and θ is a temperature difference (usually surface to free stream temperature). The transition from laminar to turbulent flow is determined by the product $(Pr \cdot Gr)$ known as the Rayleigh number, Ra. As a simple example, for plane or cylindrical vertical surfaces, it is found that

For $Ra < 10^9$, flow is laminar and $Nu = 0.59(Pr \cdot Gr)^{0.25}$

For $Ra > 10^9$, flow is turbulent and $Nu = 0.13(Pr \cdot Gr)^{1/3}$

The representative length dimension is height and the resulting heat transfer coefficients are average values for the whole height. Film temperature is used for fluid properties. Warnings similar to those given for forced convection apply to the use of these equations.

Phase change convection heat transfer (condensing and evaporation) shows coefficients that are, in general, higher than those found in single-phase flow. They are not discussed here but information may be found in standard texts.

Radiation

In space technology applications heat transfer systems are designed for conduction and radiation transfer only. When an emitting body is not surrounded by the receiver, the spatial distribution of energy from the radiating point needs to be known. To determine this distribution the intensity of radiation i_ϕ is defined in any direction ϕ as

$$i_\phi = \left(\frac{d\dot{E}''}{d\omega}\right)_\phi$$

where $d\omega$ is a small solid angle subtended at the radiating point by the area intercepting the radiation, $d\omega = dA/r^2$ (Figure 3.67) (the solid angle represented by a sphere is 4π steradians). Lambert's law of diffuse radiation states that $i_\phi = i_n \cos \phi$ where i_n is the normal intensity of radiation which can be determined for black and gray bodies;

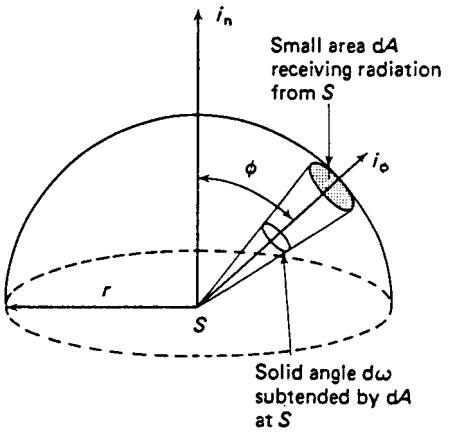

FIGURE 3.67 Spatial distribution of radiation.

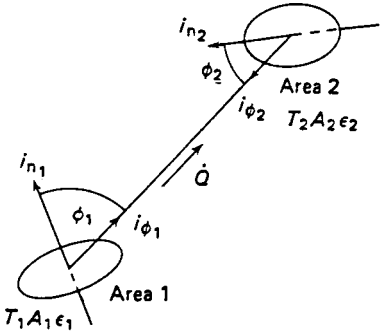

FIGURE 3.68 Heat transfer by radiation between two arbitrarily disposed gray surfaces.

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Black $i_n \sigma T^4/\pi$ Gray $i_n\ \varepsilon\sigma T^4/\pi$

With this knowledge of the radiation intensity in any direction it is only necessary to determine the amount that any body can see of any other body to calculate the heat transfer rate. For this purely geometric problem mathematical analysis (Figure 3.68) suggests a quantity variously known as the geometric, configuration or shape factor, which is defined as the fraction of the energy emitted per unit time by one surface that is intercepted by another surface. The geometric factor is given by

$$F_{12} = \frac{1}{A_1}\int_{A_1}\int_{A_2}\frac{\cos\phi_1\ \cos\phi_2\ dA_1 dA_2}{\pi x^2}$$

It can be seen that $A_1F_{12} = A_2F_{21}$, a useful reciprocal relation. It is also clear that the equation will require skill to solve in some situations, and to overcome this problem geometric factors are available for many situations in tables or on graphs (Hottel charts). The charts can give more information than anticipated by the use of shape factor algebra, which enables factors to be found by addition, subtraction, etc. (Figure 3.69).

Having established the intensity of radiation and the geometric factor, problems may be solved by an electrical analogy using the radiosity of a surface. Radiosity is defined as the total emitted energy from a gray surface:

$$\dot{J}'' = \dot{E}''_g + \rho\dot{G}''$$

where $\dot{J}''$ is the radiosity and $\rho\dot{G}''$ the reflected portion of the incident radiation $\dot{G}''$. Since $\dot{E}''_g = \varepsilon\dot{E}''_0$ the net rate of radiation leaving a gray surface of area A becomes

$$\frac{\dot{E}''_b - \dot{J}''}{\rho/A\varepsilon}$$

which may be envisaged as a potential difference, $\dot{E}''_b - \dot{J}''$, divided by a resistance, ρ/AE. A similar geometric resistance of $1/AF$ can be established to enable complete circuits to be drawn up. Thus for a three-body problem we may sketch the analogous electrical circuit (Figure 3.70) and apply Kirchhoff's electric current law to each J'' node to obtain three simultaneous equations of the form

$$\frac{\dot{E}''_{b_1} - \dot{J}''_1}{\rho_1/A_1\varepsilon_1} + \frac{\dot{J}''_2 - \dot{J}''_1}{1/A_1F_{12}} + \frac{\dot{J}''_3 - \dot{J}''_1}{1/A_1F_{13}} = 0$$

If there are more than three bodies sketching becomes complex and the equation above can be rearranged and generalized. For N surfaces ($j = 1$ to N) there will be N equations, the ith of which ($i = 1$ to N) will be

$$\dot{J}''_i - (1 - \varepsilon_i)\sum_{j=1}^{N} F_{ij}\dot{J}''_j = \varepsilon_i\dot{E}''_{b_i}$$

When $j = i$, F_{ii} will be zero unless the the surface is concave and can see itself. This set of N simultaneous equations may be solved by Gaussian elimination for which a computer program may be used. The output of the solution will be N

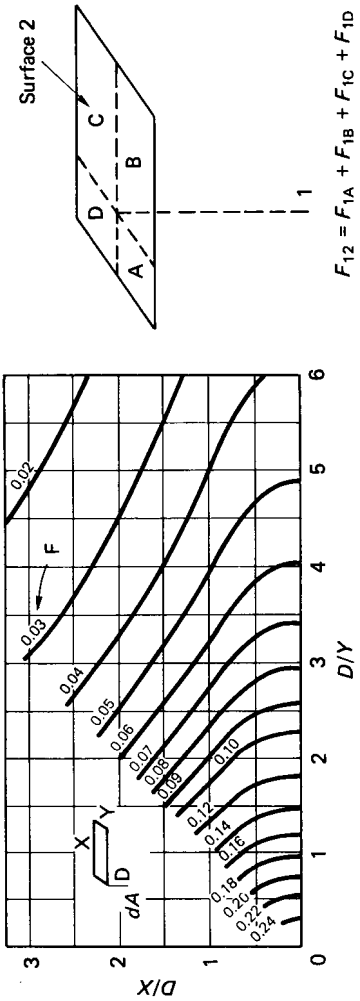

FIGURE 3.69 The geometric factor chart.

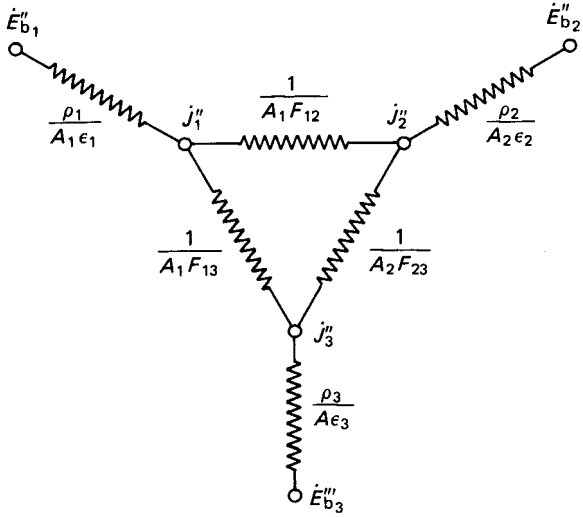

FIGURE 3.70 The electrical analogy for gray-body radiation problems.

values of $\dot{J}''$ and any required heat flux can be found. In the three-body problem, for example, the heat transfer from 1 to 2

$$\dot{Q}_{12} = \frac{\dot{J}''_1 - \dot{J}''_2}{1/A_1F_{12}}$$

or the total heat transfer from body 1 is

$$\frac{\dot{E}''_1 - \dot{J}''_1}{\rho_1/A_1\varepsilon_1}$$

Special cases are:

1. If $N - 1$ bodies are in large surroundings then for this Nth body $\dot{J}'' = \dot{E}''_b$.
2. An insulated or refractory surface has no black body potential but contributes to the heat transfer by taking up an equilibrium temperature T given by $\sigma T^4 = \dot{J}''$
3. Radiation shield problems will show six resistances in series rather than the series-parallel circuits used previously.

In all the discussion on radiation above no account has been taken of the selective emitter for which emissivity is not constant. Additional techniques are required to solve these real problems and care should be exercised if widely varying emissivity is encountered. In new situations it will be necessary to determine emissivity by experiment before proceeding. No account has been taken of intervening media for which transmissivity is not unity. Gas absorption and radiation needs further information. For gases such as oxygen, nitrogen and hydrogen with symmetric, diatomic molecules, the above work is adequate, but asymmetric molecular struc-

tures cause problems. In particular, hydrocarbon fuel combustion products (H_2O, CO_2, CO, SO_2) are important in engineering calculations and account must be taken of their radiation properties. Solar radiation problems also need special consideration.

Finned Surfaces

In many heat exchange problems involving the determination of a *U*-value it is found that the surface heat transfer coefficient on one side of the solid interface is much smaller than that on the other. The smaller coefficient will dictate the heat transfer rate achieved, and in order to overcome this problem fins may be added to this poor convection surface to increase the area for heat transfer. This problem will occur in liquid to gas exchangers on the gas side. The addition of fins will alter the flow pattern so that a new coefficient should if possible be determined. It is also possible that there may be variation of coefficient over the fin surface.

Simple fin theory in which conduction along the fin is balanced with convection from the surface can be used to determine the temperature distribution and heat transfer rate of the fin. For example, when a long fin of constant cross-sectional area is examined (Figure 3.71) it is found that the temperature distribution is

$$\frac{\theta}{\theta_0} = \frac{\cosh(m(l - x))}{\cosh(ml)}$$

and the heat transfer rate is

$$\dot{Q} = mkA\theta_0 \tanh(ml)$$

where θ is the temperature difference between fin and fluid and θ_0 is the difference at the fin root, $m = hp/kA$, p is the fin perimeter and A is the cross-sectional area.

A fin efficiency is then defined to compensate for the varying temperature difference along the fin as

$$\eta_{\text{fin}} = \frac{\text{Actual heat transfer rate}}{\text{Heat transfer rate if whole fin were at the wall temperature}}$$

which for the simple case above is $\eta_{\text{fin}} = \tanh(ml)/ml$. Fins are usually fitted in

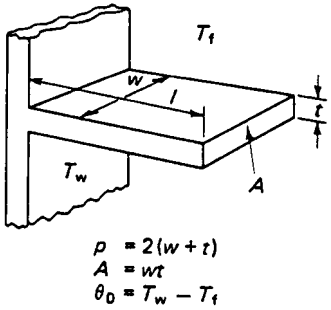

FIGURE 3.71 Simple rectangular fin nomenclature.

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arrays and the efficiency of a fin system can be established in the form of an area weighted fin efficiency, η'

$$\eta' = \eta_{\text{fin}}\beta + 1 - \beta$$

where

$$\beta = \frac{\text{total fin area}}{\text{total area including fins}} = \frac{A_{\text{fin}}}{A}$$

The U value based on the enhanced area A is then

$$\frac{1}{U_A} = \frac{1}{(A_1/A)h_1} = \sum \frac{\Delta x}{R} + \frac{1}{\eta' h_2}$$

For complete surfaces β is supplied by the manufacturer.

When fins of more complex shape are used (tapered fins or annular fins) the cross-sectional area is not constant and fin efficiency data are obtained from graphs. Care should be taken in the interpretation of such graphs since the equations above may not agree with the definitions used for the graphs.

Heat Exchangers

It is possible to design a heat exchanger with the information above and obtain a basic idea of size and configuration of simple tubular structures. It is first necessary to realize that temperature differences change along a heat exchanger, and that flow may be parallel or counter in direction. The latter is to be preferred, as it leads to smaller sizes. To allow for the changing temperature difference a log mean temperature difference is used (Figure 3.72):

$$\theta_{\text{LMTD}} = \frac{\theta_1 - \theta_2}{\ln(\theta_1/\theta_2)}$$

and to allow for varying flow patterns (which are neither counter nor parallel flow) graphs are available to give a factor F to modify the (usually) counterflow value of mean temperature difference. Thus, for any heat exchanger,

$$\theta_{\text{mean}} = F\theta_{\text{LMTD}}$$

If an estimated mean U-value for the surface is then determined the heat exchange equation is

$$\dot{Q} = UAF\theta_{\text{LMTD}}$$

Thus the area is determined. There are many solutions to this equation to satisfy all the constraints which will include:

$$\text{Energy for each stream } \dot{Q} = (\dot{m}\Delta h)_{\text{hot}} = (\dot{m}\Delta h)_{\text{cold}} \quad (\Delta h = \text{enthalpy change})$$

$$\text{Continuity of each stream } \dot{m} = \rho AV$$

$$\text{Heat transfer area } A = \pi dl$$

The above equations will involve options with varying tube numbers and di-

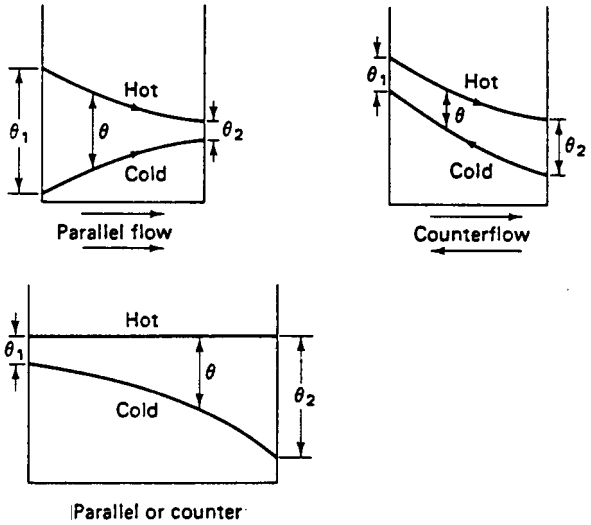

FIGURE 3.72 The mean temperature difference in heat exchangers.

ameters which will affect the determination of heat transfer coefficients (and hence the *U*-value), and a number of solutions will be obtained. The optimum choice will involve allowable pressure drops, velocities and exit temperatures. It is not a difficult calculation, but, because of the choice, a computer program may be used. There are short cuts to this approach based on the interrelation between pressure drop and heat transfer (modified Reynolds' analogy) to determine heat transfer coefficients and a method based on graphs of effectiveness (E), capacity ratio (c), and number of transfer units (NTU) is sometimes used.

By whatever method a design is achieved, it will, unless the application is very simple, be necessary to consult a professional heat exchanger designer with experience and full computer programs.

3.27 USE OF COMPUTERS

Computers may be used as an aid to heat transfer calculations at various levels of skill. At the simplest level, computer programs may be written to determine, for example, heat losses in buildings, heat inputs from pipes and radiators, etc. This will save repetitive calculations and build a small library of useful programs. To aid those whose heat transfer and computing skills are slight, Bacon (1989) gives a simple approach to problems. For more detail on the mathematics of finite difference techniques in heat transfer Myers (1971) is useful.

An alternative approach to numerical approximations in computer work is to use finite element methods. For heat transfer applications Myers (1971) is again useful. Very few people will find the need for finite element programming skills, as there

is an ever-increasing range of software aimed at the solution of many engineering problems, including heat transfer. Many of these are PC packages in which there is integration between design, drawing, manufacture and analysis of stress, vibration, heat transfer, etc.

Whenever numerical approximation techniques are used, large-scale analysis leads to considerable demand on data storage and computing time. It is therefore essential to do as much as possible with simple methods before becoming committed to large-scale finite element packages. If it is decided that the use of such a package is necessary it is vital to be sure of the requirements of the problem, for volume of input data and output results will be large and unpalatable. To assist with this problem, preprocessing packages are used for data input and mesh generation with graphic display, and post-processing packages are used for graphic display of the results. For example, the temperature distribution in a combustion chamber or piston displayed by color graphics enables easier identification of problem areas.

Finally, it must be emphasized that computing is not a substitute for understanding heat transfer problems. It is an aid to enable a more detailed investigation to be achieved and presented in a fashion to enable engineers to improve their designs.

3.28 HEAT TRANSFER: NOMENCLATURE

A	area
B	Biot number
c_p	Specific heat capacity at constant pressure
$\dot{E}''_b$	Black-body emissive power
$\dot{E}'_g$	Gray-body emissive power
F	Fourier number, geometric factor, mean temperature factor
g	gravitational constant
Gr	Grashof number
h	surface heat transfer coefficient
i_ϕ	intensity of radiation in direction ϕ
i_h	normal intensity of radiation
$\dot{J}''$	radiosity
k	thermal conductivity
l	length
m	mass, fin parameter
$\dot{m}$	mass flow rate
Nu	Nusselt number
p	perimeter
Pr	Prandtl number
Q	heat transfer
$\dot{Q}$	heat transfer rate
$\dot{Q}'$	heat transfer rate per unit length

$\dot{Q}''$	heat transfer rate per unit area
$\dot{Q}'''$	heat transfer rate per unit volume
r	radius
Ra	Rayleigh number
Re	Reynolds number
St	Stanton number
T	temperature
t	time
U	overall heat transfer coefficient per unit area
U'	overall heat transfer coefficient per unit length
V	velocity
x, y, z	rectangular coordinates

Greek Letters

α	thermal diffusivity, absorbtivity
β	coefficient of cubical expansion
Δ	change in
η	efficiency
ε	emissivity
λ	wavelength
μ	viscosity
ϕ	angle
ρ	density, reflectivity
σ	Stefan–Boltzmann constant
Σ	summation
θ	temperature difference
τ	transmissivity
φ	solid angle

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